MATHEMATICS

A Textbook for Class XI

Part I

IZHAR HUSAIN U. B. TEWARI
M.S. RANGACHARI D. D. JOSHI
V. KANNAN B. DEOKINANDAN

राष्ट्रीय शैक्षिक अनुसंधान और प्रशिक्षण परिषद्
NATIONAL COUNCIL OF EDUCATIONAL RESEARCH AND TRAINING

First Edition
August 1988 Sravana 1910

-OFFICES OF THE PUBLICATION DEPARTMENT, NCERT-

NCERT Campus	CWC Campus	Navjivan Trust Building	CWC Campus
Sri Aurobindo Marg	Chitlapakkam, Chromepet	P.O. Navjivan	32, B.T. Road, Sukchar
NEW DELHI 110016	**MADRAS 600064**	**AHMEDABAD 380014**	**24 PARGANAS 743179**

Published at the Publication Department by the Secretary, National Council of Educational Research and Training, Sri Aurobindo Marg, New Delhi 110 016, lasertypeset at Indian Institute of Science, Bangalore 560 012 and printed at Shahjahanabad City Print. Pvt. Ltd., 90, DSIDC Shed, Okhla Industrial Estate, Phase II, New Delhi 110 020

Foreword

In keeping with the National Policy on Education (NPE) 1986, the National Council of Educational Research and Training (NCERT) developed a new curriculum in mathematics covering the entire school stage. It brought out new textbooks and other related instructional materials. The textbooks were developed by teams of authors consisting of eminent mathematicians with active interaction with user teachers. These textbooks have been in use in the schools affiliated to the Central Board of Secondary Education (CBSE) since 1988. Several states have also adopted/adapted these textbooks.

The Department of Education in Science and Mathematics (DESM) of the NCERT collected a lot of information from selected schools regarding the classroom use of the materials. In addition, very useful feedback was received by the DESM as well as by the authors themselves from several quarters. In the light of the feedback obtained, it was felt that the mathematics textbooks for Classes XI and XII needed some revision. Consequently, a small group consisting of Professor M. S. Rangachari, Professor A.M. Vaidya and Professor V. Kannan was set up to revise the books. Each of these authors separately consulted the teachers teaching Classes XI and XII through special workshops and finally the books were revised. I am grateful to all of them for consenting to take up the responsibility of revising the books and doing a good job within the short time available

I also thank Prof. K. V. Rao, Dr B. Deokinandan, Shri Mahendra Shanker, Dr S. K. S Gautam, Dr Ram Avtar, Dr Hukum Singh and Shri G. D. Dhall of the DESM who, besides contributing to material development took lots of pains to see the books through the press. I am indebted to the teachers, students, parents and institutions who favoured us with valuable comments and suggestions which formed the basis for the revision.

Though some improvements have been made in the present revised books, there will always be ample scope for further improvement So, suggestions/comments from users and others are most welcome.

A. K. Sharma
Director
National Council of Educational
Research and Training

SCIENCE RELATED VALUES

Curiosity, quest for knowledge, objectivity, honesty and truthfulness, courage to question, systematic reasoning, acceptance after proof/verification, open-mindedness, search for perfection and team spirit are some of the basic values related to science. The processes of science, which help in searching the truth about nature and its phenomena are characterised by these values. Science aims at explaining things and events. Therefore to learn and practise science :

- **Be inquisitive about things and events around you.**
- **Have the courage to question beliefs and practices.**
- **Ask 'what', 'how' and 'why' and find your answers by critically observing, experimenting, consulting, discussing and reasoning.**
- **Record honestly your observations and experimental results in your laboratory or outside it.**
- **Repeat experiments carefully and systematically if required, but do not manipulate your results under any circumstance.**
- **Be guided by facts, reasons and logic. Do not be biased in one way or the other.**
- **Aspire to make new discoveries and inventions by sustained and dedicated work.**

Preface

The textbooks of mathematics for senior secondary classes developed during 1987-88 by the author-team constituted by the NCERT have been generally appreciated by their users in schools as well as by others interested in mathematics education However, in course of time, some good suggestions have been received by the authors. Also, the Department of Education in Science and Mathematics, NCERT gathered some feedback from various schools in which the textbooks have been taught.

The NCERT desired that these textbooks be revised in the light of the feedback received from several quarters and entrusted the job of revision to Prof. A.M. Vaidya (as a non-author), Prof. Kannan and myself (as among the authors of these textbooks). We three went through the textbooks critically. Further, all the three of us conducted small workshops separately inviting teachers teaching these materials in their schools and other experienced in teaching at the senior secondary and higher levels to discuss the revision. After revising the material in the workshops they were further more closely scrutinised with the help of Dr V. K Krishnan and Kum. R. Vijaylakshmi before they were passed on to the NCERT for printing.

The main features of the revision are:

(i) Change of sequence of topics/concepts in the textbook of Class XI to ensure more logical development and inclusion of historical references at the appropriate places in the text itself rather than as an appendix.

(ii) Inclusion of some additional concepts like moment of a couple, Bayes' formula, etc. either introduced by CBSE and other Boards or to plug loopholes in the earlier version of the text. (The concept of rank of a matrix is introduced to treat more precisely the consistency of a system of equations.)

(iii) Inclusion of additional exercises.

(iv) Elimination of errors that had inadvertently crept into the earlier version of the textbooks.

I am very grateful to Prof. A. M Vaidya and Prof. Kannan for the pains they have taken to complete the work in a short time in spite of their various other commitments. I must make special mention of the valuable contribution made by Dr V K. Krishnan and Kum. R. Vijaylakshmi who went through the minute details in the revised manuscript and helped in the further refinement of the manuscript. I must thank Prof. K.V. Rao, Dr B. Deokinandan and Shri Mahendra Shanker of the Department of Education in Science and Mathematics, NCERT who initiated the revision, gave all support to us and finally saw the revised books through the press. Lastly, I am very grateful to the teachers who attended the workshops for revision of the textbooks and many other friends who made valuable comments/suggestions that helped in the revision of the

textbooks. The production team of the Publication Department of the NCERT deserve all appreciation for the good quality of production and expeditious printing. In spite of all the care taken so far it is possible that there are still some shortcomings in the revised textbook. I shall be grateful for any suggestions or comments which will help the further improvement of the material.

M.S. RANGACHARI
The Ramanujan Institute
University of Madras

Contents

CHAPTER 1

Sets, Relations and Functions

1.1 Set

This chapter is intended mainly as a review of the topics discussed in the textbooks of lower classes. You have now become familiar with the concept of 'set'. As you know a set goes along with the concepts: "objects", "belongs to" or "does not belong to" the set. If S is a set and a belongs to the set S, we write "$a \in S$", which is read as 'a belongs to S' or 'a is an element of S' or 'a is in S'. In Fig. 1.1 if S is a set of points of which a is one, then $a \in S$.

If a does not belong to S, we write $a \notin S$, a is then not an element of S. In mathematics, we are mostly concerned with sets of numbers, or sets of other mathematical entities e.g. sets of polynomials, sets of fractions, sets of lines, sets of circles and so on. A rigorous theory of sets has been developed on the basis of axioms but for our purpose an intuitive approach known as "naive set theory"* would be enough. A set is determined, if, given any object, it can be said whether the object belongs to or does not belong to the set.

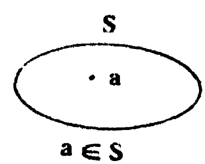


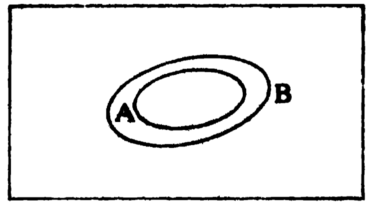
Fig.1.1

In any context, we have in mind some set and we consider different subsets of this set. This set is called the *universal set*. For example, when in two dimensional geometry we discuss sets of lines or triangles or circles, then the universal set may be the plane in which the lines, circles and triangles lie. Suppose we are discussing integers, positive integers, or prime numbers, then we can take the universal set to be $\mathbf{Z}$, the set of integers. We can also take $\mathbf{R}$, the set of real numbers, as the universal set in this case. Thus the universal set is determined by the context of the problem discussed. You also know what is meant by the empty set or the void set, or the null set which is denoted by the symbol ϕ, which is the letter 'oh' of the Scandinavian alphabet.

* A nicely written book with this title by P.R. Halmaos is recommended for further study.

If A and B are two sets such that every element of A is also an element of B, then we say that 'A is a subset of B'. In symbols, we write $A \subset B$ (See Fig. 1.2).

For example, $\mathbf{Z} \subset \mathbf{Q}$, (where $\mathbf{Q}$ denotes the set of rational numbers), since every integer is a rational number. It may be noted that $A \subset A$ and the empty set is a subset of A for every set A. If $A \subset B$, we sometimes write $B \supset A$ and read 'B contains A' or 'B is a superset of A'. We can also say 'A is contained in B'.



A ⊂ B

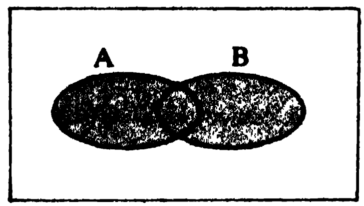
Fig 1.2

Two sets are said to be equal if they have the same elements, more precisely, elements of A are in B and *vice versa*. You can see that if $A \subset B$ and $B \subset A$, then $A = B$.

Let us recall that the union of two sets A and B is the set consisting of all the elements of A together with all the elements of B. There is no point here in repeating the elements more than once. The union of two sets A and B is written as $A \cup B$ (See Fig. 1.3).

In symbols, we have

$$A \cup B = \{x | x \in A \text{ or } x \in B\}$$

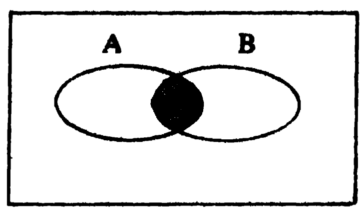


A ∪ B

Fig 1.3

The intersection of two sets A and B, denoted by $A \cap B$ is the set of elements common to A and B (See Fig. 1.4). In symbols, we have

$$A \cap B = \{x | x \in A \text{ and also } x \in B\}$$



A ∩ B

Fig 1.4

For example, if A is the set of positive integral multiples of 2 and B is the set of positive integral multiples of 3,

$$\begin{aligned} \text{i. e., } A &= \{x | x = 2n, n = 1, 2, 3, \ldots\} = \{2, 4, 6, 8, \ldots\} \\ B &= \{x | x = 3n, n = 1, 2, 3, \ldots\} = \{3, 6, 9, \ldots\} \end{aligned}$$

then

$$\begin{aligned} A \cup B &= \{x | x \text{ is a multiple of 2 } \textit{or} \text{ a multiple of 3}\} \\ &= \{2, 3, 4, 6, 8, 9, 10, 12, 14, 15, \ldots\} \end{aligned}$$

and $A \cap B = \{x | x \text{ is a multiple of both 2 and 3 }\}$
$= \{6, 12, 18, \ldots\}$
$= \{\text{all the multiples of } 6\}$

We can similarly define the union and intersection of any number of sets.

If U is the universal set and $A \subset U$, then the complement of A with respect to U denoted by A^c, is the set of all those elements of U which do not belong to A (See Fig. 1.5).

In symbols,

$$A^c = \{x | x \in U, x \notin A\}$$

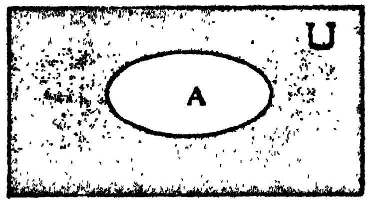


Fig 1.5

Evidently

$$(A^c)^c = A$$
$$U^c = \phi$$

De Morgan's Laws

If A and B are two subsets of U, then it can be shown that

$$(A \cup B)^c = A^c \cap B^c$$

and $$(A \cap B)^c = A^c \cup B^c$$

These relations are true even for more than two sets. These results may be verbally stated as follows:

Complement of union is equal to intersection of complements and complement of intersection is equal to union of complements.

We use the notation $\bigcup_{i=1}^{n} A_i$ to denote the union of n sets $A_1, A_2, A_3, \ldots, A_n$. Thus,

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup A_3 \ldots \cup A_n$$

Similarly, we use the notation $\bigcap_{i=1}^{n} A_i$ to denote the intersection of n sets $A_1, A_2, \ldots, A_n$. Thus,

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap A_3 \ldots \cap A_n$$

The difference of two sets A and B, denoted by $A - B$, is defined by

$$A - B = \{x | x \in A \text{ and } x \notin B\}$$

$A - B$ is also denoted by $A \setminus B$.

It may be noted that unions and intersections are commutative, associative and each of them is distributive over the other, i.e.

$$\begin{aligned}
A \cup B &= B \cup A \\
A \cap B &= B \cap A \\
(A \cup B) \cup C &= A \cup (B \cup C) \\
(A \cap B) \cap C &= A \cap (B \cap C) \\
(A \cup B) \cap C &= (A \cap C) \cup (B \cap C) \\
(A \cap B) \cup C &= (A \cup C) \cap (B \cup C)
\end{aligned}$$

Remarks

(i) The words 'family', 'class', 'collection' are also used as synonyms for the word 'set'

(ii) If $A \cap B = \phi$, we say A and B are disjoint. If $A_1, A_2 \ldots$ is a sequence of sets, it is said to be a pairwise disjoint family of sets if and only if any two sets of this family are disjoint. For example, if A is the set of all odd integers and B, that of even integers, then A and B are disjoint. The class of sets $\{A_2, A_3, A_5, A_7\}$, where A_2, A_3, A_5, A_7 are defined by

$$\begin{aligned}
A_2 &= \{2, 2^2, 2^3, 2^4, 2^5, \\
A_3 &= \{3, 3^2, 3^3, 3^4, 3^5, \\
A_5 &= \{5, 5^2, 5^3, 5^4, 5^5, \\
\text{and } A_7 &= \{7, 7^2, 7^3, 7^4, 7^5,
\end{aligned}$$

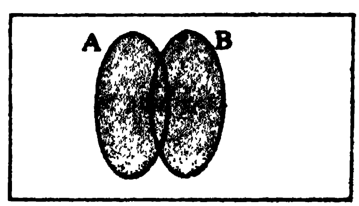
is pairwise disjoint.

Note that ϕ is such that $\phi \subset A$ for any A and $\phi \cap A = \phi$ for any A. i.e; the null set is contained in every set and at the same time disjoint from every set.

(iii) A set S is said to be a finite set if the number of elements of S is finite. The null set ϕ is regarded as a finite set

Example 1.1

Show that (i) $A \subset A \cup B$
(ii) $A \cap B \subset A$



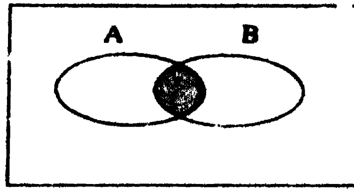
$A \subset A \cup B$

Fig 1.6

Solution

(i) Let $x \in A$. Then $x \in A$ certainly, while x may belong to B or may not belong to B. So in either case, $x \in A \cup B$. Hence, $A \subset A \cup B$. (See Fig. 1.6).

(ii) If $x \in A \cap B$, then $x \in A$ and $x \in B$. So in particular, $x \in A$. Hence $A \cap B \subset A$. Similarly $A \cap B \subset B$ (See Fig. 1.7).



$A \cap B \subset A$

$A \cap B \subset B$

Fig. 1.7

Remark

You might have used Venn diagrams in lower classes to verify set theoretic facts. It is, however, necessary to draw the diagrams in the most general way. For example, suppose we have to represent three sets. Then Fig. 1.8 does not represent the general case, since it would follow that $A \cap B \cap C = \phi$, which need not always be the case. Fig. 1.9 represents the general case.

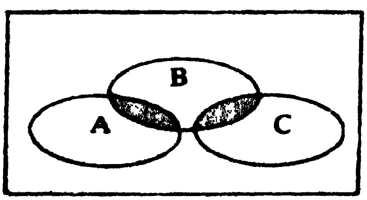


Fig. 1.8

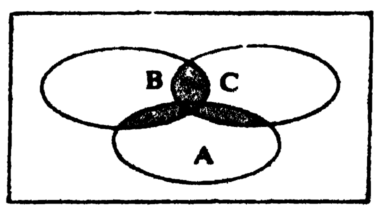


Fig. 1.9

1.2 Cartesian Product of Sets, Relations

The procedure of considering union or intersection of a class of sets and the difference between two given sets is to create more sets out of given sets. Yet another procedure is to consider what is known as the Cartesian product of two or more sets. For simplicity, let A, B be two sets. By an ordered pair of elements we mean a pair (a, b), $a \in A$, $b \in B$ in that order. (a, b), (b, a) are different unless $a = b$ i.e. a and b are one and the same object. The set of all ordered pairs (a, b), of elements $a \in A$, $b \in B$ is called the ***Cartesian product*** of the sets A and B and is denoted by $A \times B$. In symbols,

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

We say $(a,b) = (c,d)$ if and only if $a = c$ and $b = d$. Clearly, $A \times B$ and $B \times A$ are different sets if $A \neq B$. Also $A \times B = \phi$ when one or both of A, B are empty. Conversely if $A \times B = \phi$, either $A = \phi$ or $B = \phi$. Again, $A \times B = B \times A$ if and only if $A = B$ or $A = \phi$ or $B = \phi$.

If there are 3 sets X, Y, Z, then choosing 3 elements x,y and z such that $x \in \mathbf{X}$, $y \in \mathbf{Y}$ and $z \in \mathbf{Z}$, we form an ordered triplet (x, y, z). The set of all such ordered triplets is called the Cartesian product of the three sets X, Y, Z and is denoted by $X \times Y \times Z$. We can, similarly, form the Cartesian product of n sets. Clearly, each element of the Cartesian product of n sets $A_1, A_2, \ldots, A_n$ is an ordered n-tuple $(a_1, a_2, a_3, \ldots, a_n)$ where $a_1 \in A_1$, $a_2 \in A_2$ and so on. The Cartesian product of n sets $A_1, A_2, \ldots, A_n$ is denoted by $A_1 \times A_2 \times \quad \times A_n$ or briefly by $\prod\limits_{i=1} A_i$.

Example 1.2

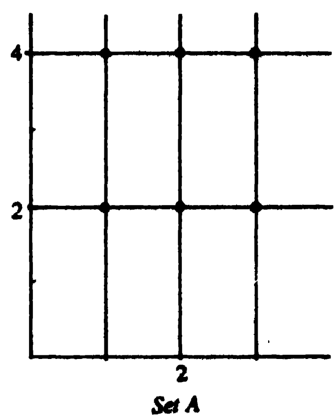
Let $A = \{1,2,3\}$ and
$B = \{2,4\}$
Find $A \times B$ and show it graphically.

Solution

Clearly $A \times B = \{(1,2),(1,4),(2,2),(2,4),(3,2),(3,4)\}$

It may be guessed from the above example that $n(A \times B) = n(A) \times n(B)$, where $n(A)$ and $n(B)$ denote the number of elements of A and B respectively and $n(A \times B)$ denotes the number of elements of the set $A \times B$.

To show $A \times B$ graphically, we draw two perpendicular lines, one horizontal and the other vertical. On the horizontal line, we represent the elements of A and on the vertical line, the elements of B(Fig. 1.10).



$A \times B$

Fig.1.10

If $a \in A$, $b \in B$, we draw a vertical line through a and a horizontal line through b. They will meet in a point which will denote the ordered pair (a, b). The set of points so obtained graphically represents $A \times B$.

In particular, if A is the set of all real numbers, we can deem A to consist of all points in a line. $A \times A$ will then consist of all points in the plane. If P is a point in the plane, then a and b in the corresponding ordered pair (a, b) are called the coordinates of P.

We now consider a set B of persons, as follows:

$$B = \{\text{Asha}, \text{Zarina}, \text{Mary}, \text{Sushma}\}$$

Let us consider another set A of their brothers as follows:

$$A = \{\text{Ram,Kedar,Jamil,Albert}\}$$

There is a relation 'is a brother of' between the elements of the sets A and B. If we write R for the relation "is a brother of" and if Asha has two brothers Ram and Kedar, Zarina has a brother Jamil and Mary has a brother Albert, then the above information can be represented as:

Ram R Asha, Kedar R Asha, Jamil R Zarina, Albert R Mary.

Omitting the letter R between the pairs of names and writing the pair of names as an ordered pair, the above information can also be written as a set of ordered pairs R where

$$\begin{aligned} R &= \{(\text{Ram,Asha}),(\text{Kedar,Asha}),(\text{Jamil,Zarina}),\ (\text{Albert,Mary})\} \\ &= \{(x, y) | x \in A, y \in B, xRy\} \end{aligned}$$

Thus we see that the relation "is a brother of" from set A to set B gives rise to a subset R of $A \times B$ such that $(x, y) \in R$ if and only if xRy. Let us consider another example. Let N be the set of natural numbers. Consider the relation 'has as its square' from the set N to N. If we write R for 'has as its square', then we get the statements:

$$1R1, 2R4, 3R9, 4R16\ldots$$

Again, we omit R between the pairs of numbers and write them as ordered pairs. We thus find that the relation 'has as its square' gives rise to a set R of ordered pairs where

$$\begin{aligned} R &= \{(1,1),(2,4),(3,9),(4,16),(5,25)\ldots\} \\ &= \left\{(x, y) | x, y \in \mathrm{N} \text{ and } y = x^2\right\} \end{aligned}$$

The set **R** obtained from the relation 'has as its square' from N to N is subset of $\mathrm{N} \times \mathrm{N}$. Again we note that $(x, y) \in R$ if and only if $y = x^2$.

Keeping the above examples in the background, we can now define a relation.

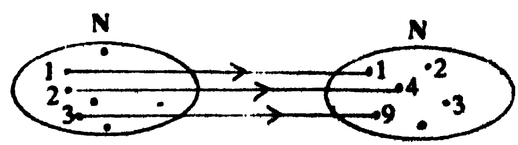


Fig.1.11

Definition

A relation R from a set A to a set B is a subset of $A \times B$. A is called the *domain* of R and B is called the *co-domain* of R.

The set of second entries of the ordered pairs in a relation is called the *range* of the relation.

Then in the example in Fig. 1.11

$$\begin{aligned} \text{domain} &= \{1, 2, 3, \ldots\} \\ \text{and range} &= \{1, 4, 9, \ldots\} \end{aligned}$$

If $(a, b) \in R$, then we write it as aRb and read it as 'a is in relation R to b'. If $A = B$, then the relation is called a relation defined in A or simply a relation in A.

A relation R in A is said to be *reflexive* if aRa for all $a \in A$. It is said to be *symmetric* if aRb implies bRa. It is said to be *transitive* if aRb and bRc together imply aRc. Let a, b be two triangles in a class of all triangles in a plane and let aRb be "a is congruent to b". Then we know that R is a relation which has all the properties mentioned above. Any such relation which is reflexive, symmetric and transitive is called an *equivalence relation.*

Not all relations are equivalence relations. In the set N of natural numbers aRb $a, b \in N$ defined by 'a divides b' or in symbols, $a|b$, is a reflexive relation but not symmetric. It is, however, transitive. So it is not an equivalence relation.

An important property of an equivalence relation is that it divides the set into pairwise disjoint subsets whose collection is called a *partition* of the set. Note that the union of the subsets in the collection is the whole set. We illustrate this point by the following example.

In the set N of natural numbers, we define a relation R as follows:

For $n, m \in \mathrm{N}$, nRm if on division by 5 each of the integers n and m leaves the same remainder i.e. one of the numbers 0,1,2,3 and 4. It is easily seen that R is an equivalence relation. For, aRa for all $a \in \mathrm{N}$ (Reflexive). If aRb, then bRa for $a, b \in \mathrm{N}$ (Symmetric). If aRb, and bRc, then aRc (Transitive).

Let $A_o = \{\ n | n \in \mathrm{N}$ and on division by 5, n leaves the remainder 0 $\}$

$A_1 = \{\ n | n \in \mathrm{N}$ and on division by 5, n leaves the remainder 1 $\}$

Similarly, we define sets A_2,A_3, and A_4. Since there can be only five remainders viz. 0,1,2,3,4 on division by 5, we shall not get any other sets.

Now,

$$\begin{aligned}
A_0 &= \{5, 10, 15, 20, \ldots\} \\
A_1 &= \{1, 6, 11, 16, 21, \ldots\} \\
A_2 &= \{2, 7, 12, 17, 22, \ldots\} \\
A_3 &= \{3, 8, 13, 18, 23, \ldots\} \\
A_4 &= \{4, 9, 14, 19, 24, \ldots\}
\end{aligned}$$

It is evident that the above five sets are pairwise disjoint and

$$A_0 \cup A_1 \cup A_2 \cup A_3 \cup A_4 = \bigcup_{1=0}^{4} A_i = \mathrm{N}$$

We have thus seen that the equivalence relation R, defined in this example, has divided the set N into five pairwise disjoint subsets.

Conversely, a partition of a set defines an equivalence relation. If $S_1, S_2, \ldots S_n$ is a partition, this equivalence relation is aRb if and only if $a, b \in S_i$ for some $i = 1, 2, \ldots, n$.

Example 1.3

If $A = \{1, 2\}, B = \{3, 4\}, C = \{4, 5\}$, what is $A \times (B \cup C)$?

Solution

$B \cup C = \{3, 4, 5\}$ So, $A \times (B \cup C) = \{(1,3), (1,4), (1,5), (2,3), (2,4), (2,5)\}$

Example 1.4

Prove that

$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

Solution

If $(x, y) \in A \times (B \cap C)$, then $x \in A$ and $y \in B \cap C$.
So, $x \in A, y \in B$ and $x \in A, y \in C$ i.e. $(x, y) \in A \times B$ and $(x, y) \in A \times C$
Hence, $(x, y) \in (A \times B) \cap (A \times C)$
Hence, $A \times (B \cap C) \subset (A \times B) \cap (A \times C)$ (i)
Conversely, if $(x, y) \in (A \times B) \cap (A \times C)$, then $(x, y) \in A \times B$ and $(x, y) \in A \times C$.
So, $x \in A$, and $y \in B$ and $y \in C$ i.e. $x \in A$ and $y \in B \cap C$.
Hence, $(x, y) \in A \times (B \cap C)$
Hence, $(A \times B) \cap (A \times C) \subset A \times (B \cap C)$ (ii)
From (i) and (ii), we get

$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

Example 1.5

If $a, b \in N$ and R is "a is a divisor of b", then R is a relation on N. The subset S of $N \times N$ which corresponds to the relation is

$$S = \{(n, rn) : n \in N, r \in N\}$$

For instance (1,3),(3,15), (4,4) are in S while (2,5),(3,7) do not belong to S.

Example 1.6

If $a, b \in \mathbf{R}$, the set of all real numbers and R is "$| a - b |$ is a rational number", then R is a relation on $\mathbf{R}$. The subset of $\mathbf{R} \times \mathbf{R}$ which define the relation is

$$S = \{(a, \underline{a} + a) : a \in R, \underline{a} \in Q, \text{ the set of all rational numbers } \}$$

Here $(1, 2\frac{1}{2}), (-\sqrt{2}, \frac{3}{2} - \sqrt{2}), (\pi, \pi - \frac{1}{2})$
are in S. $(\sqrt{2}, \pi + \sqrt{2}) \notin S$.

Example 1.7

If $a, b \in \mathbf{N}$ and aRb if "$b-a$ is divisible by a fixed number $m \in \mathbf{N}$" then R is a relation on $\mathbf{N}$. The subset S of $N \times N$ corresponding to the relation is

$$S = \{(n, n+rm) : n \in N, r \in \mathbf{N}\}$$

$$\text{If } m = 3, (2,8), (5,11) \in S \text{ while } (3,8) \notin S.$$

Note: A relation S which is a subset of $A \times B$ may not be such that the ordered pairs in S exhaust A in their first entries.

1.3 Functions

The concept of a function is a special case of that of a relation. To be specific, while a relation may relate an element of the domain to more than one element of the range, a function relates each element of the domain to one and only one element of another set viz. the co-domain. In other words, a function is a single-valued association of all the elements of the domain with elements of the co-domain. Thus, if $\mathbf{R}^+$ denotes the set of all non-negative real numbers, f, defined by

$$f(x) = \text{ square root of } x, x \in \mathbf{R}^+,$$

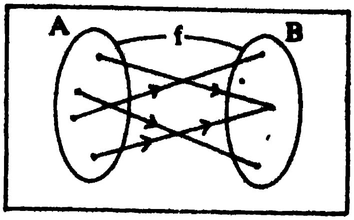
defines only a relation while the definition

$$f(x) = \text{ non-negative square root of } x, x \in \mathbf{R}^+,$$

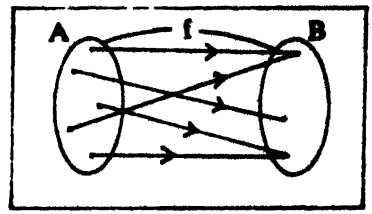
gives a function.

If $f : A \to B$ is a function, by the *graph of* f is meant the subset $\{(a, f(a)) | a \in A\}$ of $A \times B$. Two functions $f, g : A \to B$ are equal (or the same) if and only if for each $a \in A, f(a) = g(a)$. This is equivalent to saying that the graph of f and the graph of g are one and the same set. We need not, therefore, distinguish a function from its graph. If $f : \mathbf{R} \to \mathbf{R}$, we note that the graph of f is precisely the graph in the usual sense with reference to a pair of rectangular axes. Hence the terminology. Incidentally, the graph of a function $f : A \to B$ is precisely the subset of $A \times B$ which is determined by the relation which is given by the function f.

Let $f : A \to B$ be a function. Then $A = dom f$ and $B = codom f$. f is said to be a function defined on A (or map of A) *into* B(See Fig. 1.12).



Into function
Fig 1.12



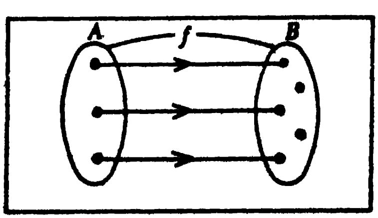
Onto function (surjection)
Fig.1.13

If the range of f i.e. $ranf$, is such that $ranf = B$, f is said to be a function defined A (or map of A) *onto* B (See Fig. 1.13). Note that $f : A \to B$, which is *onto*, is into B.

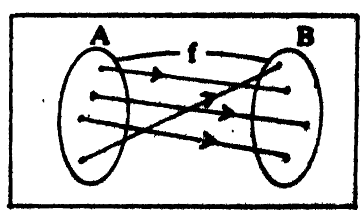
Such a function is also sometimes said to be *surjective*. If distinct elements of A are taken to distinct elements of B by f, i.e.

If $x_1, x_2 \in A, x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$, f is said to be *one-to-one* or, sometimes, *injective* function or map (See Fig. 1.14).

A map (or function) which is both one-to-one (injective) and onto (surjective) is said to be a *bijective* map (or function) or simply a *bijection* (See Fig. 1.15).

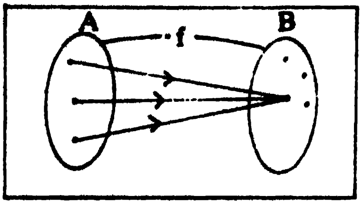


One-one into function (injective)
Fig 1.14



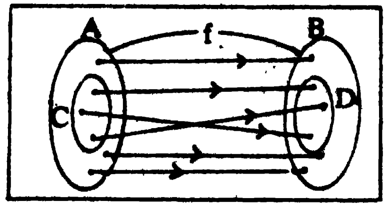
One-one onto function (bijection)
Fig.1.15

If $f(a) = b$ for every $a \in A$ and for a fixed $b \in B$, f is said to be *constant* map (See Fig. 1.16). A constant map cannot obviously be one-to-one, if its domain has more than one element. If $f : A \to B$ is a map and $C \subset A$, then we write $D = f(C) = \{b | b \in B, b = f(c) \text{ for some } c \in C\}$ and call $f(C)$ the *image* of C by f (See Fig. 1.17).



Constant function

Fig.1.16



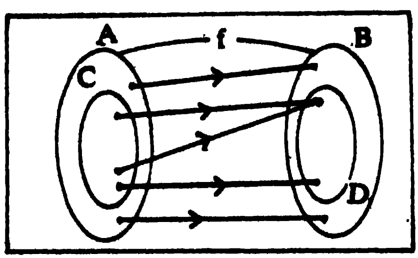
$D = f(C)$, image of C by f

Fig.1.17

In this notation $f(A) = ran f$. f is a map of A onto B if and only if $f(A) = B$. If $D \subset B$, then we write

$$f^{-1}(D) = \{a | a \in A, f(a) = d \text{ for some } d \in D\}$$

and call $f^{-1}(D)$ the *pre-image, total original or pull back* of D (See Fig. 1.18).



Pre-image of D $C = f^{-1}(D)$

Fig.1.18

A word of caution about the notation $f^{-1}(D)$. This should not be compared with the notation $f(C)$ as if there were a function f^{-1} (always). Again, the definiton of $f^{-1}(D)$ does not preclude there being an element $d \in D$ for which there exists no $a \in A$ such that $f(a) = d$. For that matter $f^{-1}(D)$ can be empty too. For instance, if $A = B = \mathbb{R}$ and $f : A \to B$ is defined by $f(a) = [a]$, the largest integer less than or equal to $a \in A$, and $D = \{\frac{1}{2}, \frac{3}{4}\} \in \mathbb{R}$, then $f^{-1}(D) = \phi$.

Let us now consider a bijection $f : A \to B$. It is then clear that for any $b \in B$ there exists one and only one $a \in A$ such that $f(a) = b$. Define the association f^{-1} of elements of B with elements of A as follows:

$$f^{-1}(b) = a \text{ if and only if } f(a) = b.$$

The observation made just above shows that this association is a function or map; viz. $f^{-1} : B \to A$. It is easy to verify that f^{-1} is a bijection too. The function f^{-1} is called the inverse of the function f (See Fig. 1.19 (a) and (b)).

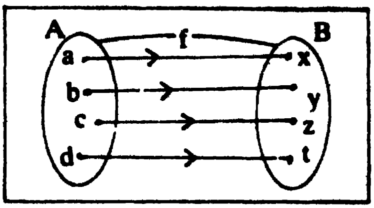


Fig.1.19 (a)

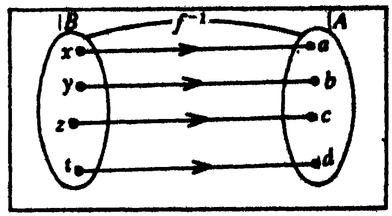


Fig.1.19 (b)

To sum up, any bijection of a set onto another has *inverse* which is also a bijection.

It is not difficult to see that $(f^{-1})^{-1} = f$. Let $f : A \to B$, $g : B \to C$, be two functions. Then the function $g \circ f$ defined by

$$(g \circ f)(a) = g\,(f(a))\,, a \in A.$$

$g \circ f : A \to C$ is called the *composition* of f and g (See Fig. 1.20).

For example, if $A = B = \mathbf{R}$ and $C = \mathbf{Z}$, the set of all integers, f, g defined respectively by

$$\begin{aligned} f(x) &= x^2, x \in \mathbf{R}, \\ g(y) &= [y]\,, y \in \mathbf{R}. \end{aligned}$$

have for their composition $g \circ f$ defined by

$$g \circ f(x) = [x^2]\,, x \in \mathbf{R}.$$

The *identity map of a set* A into itself, denoted by I_A, is defined by

$$I_A(a) = a, a \in A.$$

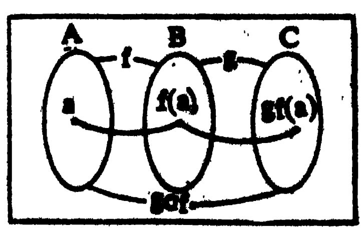


Fig.1.20

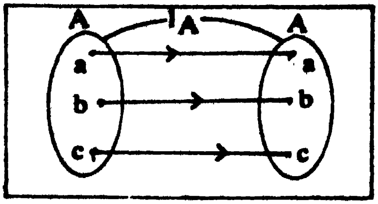


Fig.1.21

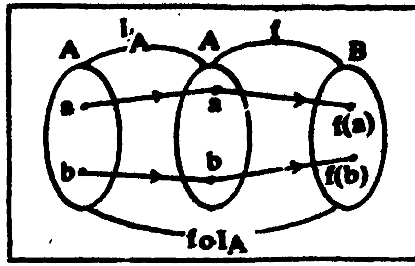


Fig.1.22

I_A is, clearly, a bijection and is its own inverse (See Fig. 1.21).
$f : A \to B$ is map, then it follows that $f \circ I_A = f$ (See Fig. 1.22).
Similarly, $I_B \circ f = f$.
In particular, if $f : A \to A$ is a map (called by some authors a self map), then

$$f \circ I_A = I_A \circ f = f.$$

Moreover, for any two maps $f, g : A \to A$, $g \circ f$ is defined and $g \circ f : A \to A$. If now, $f : A \to B$ is a bijection so that it has inverse f^{-1}, then

$$f^{-1} o f = I_A \text{ and } f \circ f^{-1} = I_B$$

If, in particular, $A = B$, so that f is a bijection of A onto itself, then

$$f^{-1} o f = f \circ f^{-1} = I_A$$

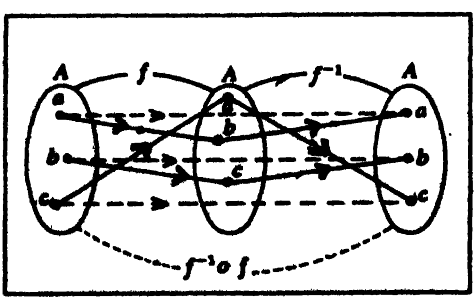


Fig.1.23

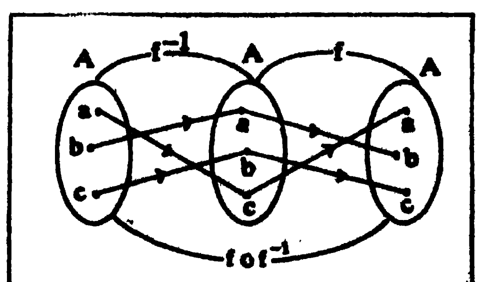


Fig.1.24

(See Fig. 1.23 and 1.24). Moreover, if $f : A \to A$, $g : A \to A$ are mappings which are such that

$$f \circ g = g \circ f = I_A$$

(See Fig. 1.25) then f, g are bijections which are inverse to each other. Justify this assertion.

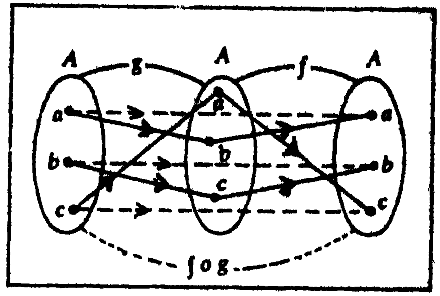


Fig.1.25

Remarks

At this stage we can rigorously define a set to be *finite* if there exists a bijection of the set on to the set $N_n = \{1, 2, 3, \ldots, n\}$ for some natural number n. The *void* set ϕ is taken to be finite.

Example 1.8

If $A = \{1, 2, 3\}$ and f, g are relations corresponding to the subsets of $A \times A$ indicated against them, which of f, g is a function? Why?

$$\begin{aligned} f &= \{(1,3),(2,3),(3,2)\} \\ g &= \{(1,2),(1,3),(3,1)\} \end{aligned}$$

Solution

f is a function since each element of A in the first place in the ordered pairs goes with only one element of A in the second place. g is not a function because 1 is related to both 2 and 3.

Example 1.9

If $f = \{(5,2),(6,3)\}$, $g = \{(2,5),(3,6)\}$, what is the range of f and g? Find $f \circ g$.

Solution

$$\begin{aligned} ran\ f &= \{2,3\}, ran\ g = \{5,6\} \\ f \circ g(2) &= f(g(2)) = f(5) = 2 \\ f \circ g(3) &= f(g(3)) = f(6) = 3 \\ \text{Hence, } f \circ g &= \{(2,2),(3,3)\} \end{aligned}$$

1.4 Binary Operations

A function $f : A \to A$, where A is a set can also be thought of as a *unitary operation* in the sense that an element of A(co-domain) is associated to each singleton subset of A (domain). If an element of A (co-domain) is associated uniquely with every subset of two elements of A (domain), the order of the elements being taken into account, we obtain a *binary operation* on A, i.e. a map $A \times A \to A$ is called a binary operation in A. Formally, for $n = 1, 2, \ldots$ and n-ary operation on the set A is a map $f : A \times A \times \ldots \times A$ (n *times*) $= A^n \to A$. For simplicity, we consider here unitary and binary operations only. If $A = \mathbf{R}^+$, the set of all positive real numbers, taking reciprocals, or, what is the same, the map $x \to \frac{1}{x} : A \to A$ is a unitary operation.

If $A = \mathbf{R}$, the set of all real numbers, $(x, y) = x + y : \mathbf{R}^2 \to \mathbf{R}$, or, what is the same, addition of two real numbers, is a binary operation on $\mathbf{R}$. Multiplication is also a binary operation on $\mathbf{R}$. However, division is not a binary operation on $\mathbf{R}$, since division by 0 is not defined. But division is a binary operation on $\mathbf{R} \setminus \{0\}$. We can think of other binary operations in terms of these known binary operations. For instance, $(m, n) \to m + n + mn : \mathbf{Z}^2 \to \mathbf{Z}$, $\mathbf{Z}$ being the set of all integers, is a binary operation on $\mathbf{Z}$. Keeping the map involved in the background, we prefer to speak of the binary operation '+' or addition instead of the map $(x, y) \to x + y$ or of the binary operation '$\cdot$', or multiplication instead of the map $(x, y) \to xy$. With an analogous notation we can speak of the binary operation '$\circ$' by writing $mon = m + n + mn$ on $\mathbf{Z}$.

We pointed out that when defining a binary operation the order of the elements is to be taken into account; in other words, the map which defines the binary operation on A is on the set A^2 of all *ordered* pairs of elements of A. If $\circ$ is the operation, for $a, b \in A$, aob and boa may be different elements of A. If, however, for every pair a, b of elements A.

$$a \circ b = b \circ a,$$

then the binary operation is *commutative*. For example, the binary operations $+, \cdot$, on $\mathbf{R}$ are commutative. Is the operation $\circ$ defined at the end of the last paragraph commutative? The operation of division in $\mathbf{R} - \{0\}$ is evidently *not* commutative. If the binary operation $*$ on $\mathbf{Z}$ is defined by

$$m * n = m - n + mn$$

then $1 * 2 = 1 - 2 + 1 \cdot 2 = 1$ while $2 * 1 = 2 - 1 + 2 \cdot 1 = 3$ so that $1 * 2 \neq 2 * 1$ and $*$ is *not* a commutative binary operation.

If $\circ$ is a binary operation on a set A and a, b, c are three elements of A, with due regard to the order in which a,b,c occur, we can consider the two elements

$$ao(boc), (aob)oc.$$

There is no *prima facie* reason for these to be one and the same. If, however, they are one and the same for *every* ordered triplet a, b, c of elements of A, then $\circ$ is said to be an *associative* operation. As examples of associative binary operations, we have addition

and multiplication defined on $\mathbf{R}$. Clearly, division on $\mathbf{R}-\{0\}$ is not an associative operation (Why?). The operation $*$ defined in the preceding paragraph which is not commutative is not also associative. If X is the set$\{a,b\}$ let the binary operation $\circ$ be defined by

$$a \circ a = a, b \circ b = b, a \circ b = b, b \circ a = a. \tag{1.1}$$

This operation is associative but *not* commutative (Verify). If the binary operation $\circ$ is defined on Z by

$$\ell \circ m = \frac{\ell + m}{2}, \ell, m \in Z,$$

the operation $\circ$ is evidently commutative. It is easily verified that this operation is *not*, however, associative. To sum up, the concepts of associativity and commutativity of a binary operation are independent.

If $\circ$ is a binary operation on X and if there is $e \in X$ such that $a \circ e = e \circ a = a$, e is said to be an *identity element* for the operation. For instance, for the binary operation of addition in $\mathbf{R}$, 0 is the identity element. For multiplication it is 1.

Remark

It is sometimes convenient to write down a binary operation by means of a table. For instance, the operation defined by (1.1) above is written in the form:

o	a	b
a	a	b
b	a	b

The presence of one or more binary operations in a set gives a structure for the set which could be studied. For instance, we have because of the availability of such operations, the concepts of group, ring, vector space, field, etc. The study of these structures is generally known as algebra.

Example 1.10

In the set $\mathbf{N}$ of natural numbers, define the binary operation $\circ$ by

$$m \circ n = \text{g.c.d}\ (m, n), m, n \in \mathbf{N}.$$

Is the operation commutative, associative?

Solution

The operation is clearly commutative since

$$\text{g.c.d}\ (m, n) = \text{g.c.d}\ (n, m)$$

It is also associative because for $\ell, m, n \in \mathbf{N}$,

$$\text{g.c.d}\ (\ell, \text{g.c.d}\ (m, n)) = \text{g.c.d}\ ((\text{g.c.d}\ (\ell, m), n))$$

Example 1.11

Let $\mathbf{N}^k$ be the set of all ordered k-tuples of natural numbers. If $x = (x_1, \ldots, x_k)$, $y = (y_1, \ldots, y_k)$, $x_i, y_i \in \mathbf{N}, i = 1, 2, \ldots, k$ define $x + y = (x_1 + y_1, \ldots, x_k + y_k)$. Then $+$ is a commutative and associative binary operation in $\mathbf{N}^k$.

Solution

It is easily verified that these properties are carried over from the corresponding properties of addition in $\mathbf{N}$.

EXERCISE 1.1

1. Prove that $A^c - B^c = B - A$

2. Prove that $A \cap (B - C) = (A \cap B) - (A \cap C)$

3. If R is the relation "less than" from $A = \{1, 2, 3, 4, 5\}$ to $B = \{1, 4, 5\}$, write down the set of ordered pairs corresponding to R. Find the inverse relation to R.

4. If R is the relation in $\mathbf{N} \times \mathbf{N}$ defined by $(a, b)R(c, d)$ if and only if $a + d = b + c$, show that R is an equivalence relation.

5. If $\mathbf{N}_7 = \{1, 2, 3, 4, 5, 6, 7\}$, which of the following two is a partition giving rise to an equivalence relation? Why?

   (i) $A_1 = \{1, 3, 5\}$, $A_2 = \{2\}$, $A_3 = \{4, 7\}$

   (ii) $B_1 = \{1, 2, 5, 7\}$, $B_2 = \{3\}$, $B_3 = \{4, 6\}$

6. If $f : A \to B$, $g : B \to C$, are one-to-one or injective functions, show that $g \circ f$ is also one-to-one.

7. If $A = \{a, b, c, d\}$ and f corresponds to the Cartesian product $\{(a, b), (b, d), (c, a), (d, c)\}$ show that f is one-to-one from A onto A. Find f^{-1}.

8. Define the binary operation $\circ$ in $\mathbf{N}$ by
   $m \circ n = \text{lcm } (m, n), m, n \in \mathbf{N}$.
   Is the operation commutative and /or associative?

9. Does the table below give a commutative binary operation on the set $\{a, b, c\}$?

o	a	b	c
a	b	c	a
b	c	a	b
c	a	b	c

10. Establish the De Morgan's laws stated in section 1.1

11. (i) Prove that if a set has only one element, then it has 2 subsets.
    (ii) If $B \subset A$ and if A has one element more than B, prove that A has twice as many subsets as B.
    (iii) Deduce from these two results that a set with 2 elements has 2^2 subsets, a set with 3 elements has 2^3 subsets and so on. How many subsets does a set with n elements have?

12. Give an example of a map

    (i) which is one-to-one but *not* onto,

    (ii) which is *not* one-to-one but onto,

    (iii) which is neither one-to-one *nor* onto.

13. If A is a non-empty set and $f, g : A \to A$ are such that $f \circ g = g \circ f = I_A$, show that f and g are bijections and that $g = f^{-1}$.

14. For any relation R in a set A, we can define the *inverse* relation R^{-1} by $aR^{-1}b$ if and only if bRa. Prove that
    R is symmetric if and only if $R = R^{-1}$

15. In $\mathbf{N} \times \mathbf{N}$, show that the relation defined by $(a, b)R(c, d)$ if and only if $ad = bc$ is an equivalence relation.

16. If $f : \mathbf{R} \to \mathbf{R}$ is defined by $f(x) = x^2 - 3x + 2$, find $f(f(x))$.

17. $f, g : \mathbf{R} \to \mathbf{R}$ are defined respectively by $f(x) = x^2 + 3x + 1, g(x) = 2x - 3$, find

    (i) $f \circ g$

    (ii) $g \circ f$

    (iii) $f \circ f$

    (iv) $g \circ g$

18. What is the set $\{x | x \in \mathbf{R}, x^2 = 9 \text{ and }, 2x = 4\}$?

19. Is inclusion of a subset in another, i.e., $A\mathcal{R}B$ if and only if $A \subset B$, in the context of a universal set, an equivalence relation in the class of subsets of the universal set? Justify your answer.

20. How many relations are possible from a set A of m elements to another set B of n elements? Why?

21. If $A = \{1, 2, 3\}$, $B = \{4\}$, $C = \{5\}$, then verify that

    (i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

    (ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

    (iii) $A \times (B - C) = (A \times B) - (A \times C)$

22. '$*$' is a binary operation defined on $\mathbf{Q}$. Find which of the binary operations are commutative

    (i) $a * b = a - b$ for $a, b \in \mathbf{Q}$

    (ii) $a * b = a^2 + b^2$ for $a, b \in \mathbf{Q}$

    (iii) $a * b = a + ab$ for $a, b \in \mathbf{Q}$

    (iv) $a * b = (a - b)^2$ for $a, b \in \mathbf{Q}$

23. '$*$' is a binary operation defined on $\mathbf{Q}$. Find which of the binary operations are associative

    (i) $a * b = a - b$ for $a, b \in \mathbf{Q}$

    (ii) $a * b = \frac{ab}{4}$ for $a, b \in \mathbf{Q}$

    (iii) $a * b = a - b + ab$ for $a, b \in \mathbf{Q}$

    (iv) $a * b = ab^2$ for $a, b \in \mathbf{Q}$

CHAPTER 2

Trigonometric Functions

2.1 Introduction

We shall now begin the study of trigonometry. It is convenient to use trigonometry to measure distances between two landmarks or widths or depths of rivers or heights of mountains, etc.

Trigonometry means the science of measuring triangles. Given some of the sides and angles of a triangle, trigonometry helps us to calculate the remaining sides and angles. The congruence results of geometry tell us that two sides and the included angle (SAS) completely determine the triangle. That is, if we know two of the three sides and the included angle of a triangle, the remaining one side and two angles become fixed. So we should be able to calculate these, but how? School geometry does not tell us how. We have to study trigonometry to be able to do that. In the same way, given three sides of a triangle (SSS), trigonometry will show us how to calculate the angles, etc.

You will be happy to know that the study of trigonometry was first started in India. Elements of the subject can be found even in Rigveda. All the ancient Indian Mathematicians like Aryabhata, Bhaskara I and II and Brahmagupta got important results. All this knowledge first went from India to middle-east and from there to Europe. The Greeks had also started a study of trigonometry but their approach was so clumsy that when the Indian approach became known, it was immediately adopted throughtout the world.

2.2 Angles

An angle is considered as the figure obtained by rotating a given ray about its end-point. The original ray is called the *initial side* and the ray into which the initial side rotates is called the *terminal side* of the angle. The *sense* of an angle is derived from the direction of rotation of the initial side into the terminal side. If this direction is counter-clockwise, the sense of the angle is said to be positive and if this direction is clockwise then the sense is negative.

The measure of an angle is the amount of rotation required to get to the terminal side from the initial side. We place the vertex of the angle at the centre of a circle of some

fixed radius. Divide the circumference of the circle into 360 equal parts, called degrees.

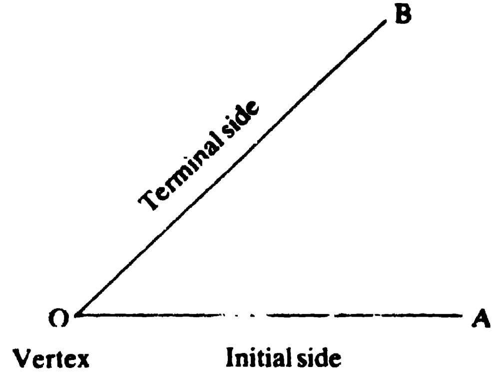


Fig.2.1

The number of degrees on the circumference between the initial and terminal sides of the angle is its degree measure. For additional precision, each degree is subdivided into 60 equal parts, called minutes and each minute is divided into 60 equal parts called seconds. The symbol $^\circ$ is used to denote degrees, $'$ is used to denote minutes and $''$ is used to denote seconds.

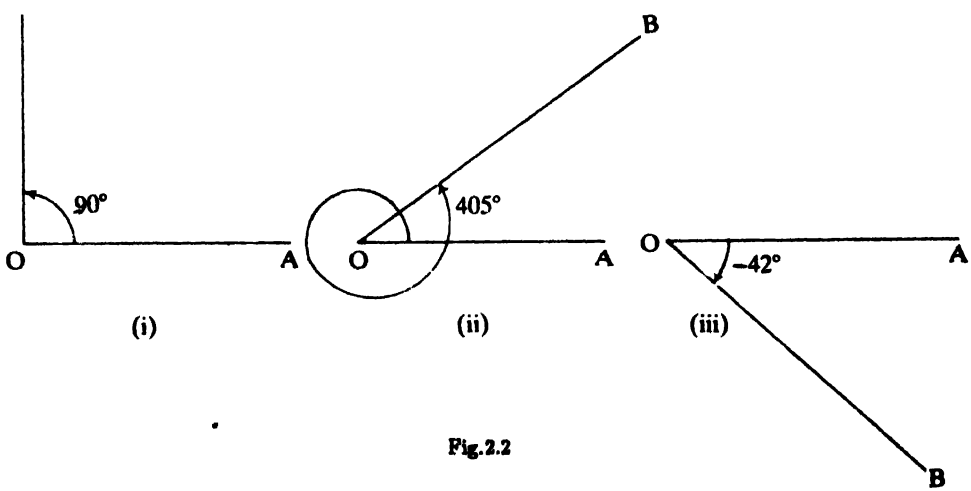


Fig.2.2

Remark

In our discussion in the above paragraph we considered a circle of fixed radius. However, the measure of an angle does not depend on the radius of the circle.

Some of the angles are shown in the following figure. Note that the terminal side has to be rotated so as to have more than one revolution if the corresponding angle is greater than 360°.

Radian Measure

There is another unit of angular measurement called the *radian measure*, which is of particular importance in higher mathematics and its applications. This is based upon the fact that the ratio of the circumference of circle to its diameter is constant. This constant π is an irrational number having the non-recurring decimal expansion $\pi = 3.14159\ldots$; 22/7 is taken as an approximate value of π.

As shown in Fig. 2.3 we place the vertex of the angle at the centre of the circle of radius r. If the length of the arc subtending the angle at the centre is s, the *radian measure* t of the angle is defined to be $\frac{s}{r}$. Note that the length of the arc is taken to be negative if we measure it in the clockwise direction. We again remark that the measure of an angle is independent of the radius of the circle considered because of the fact that the ratio of the circumference of a circle to its diameter is constant.

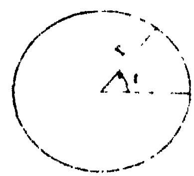


Fig.2.3

The radian measure t of the angle formed by one complete revolution of the initial side is $\frac{2\pi r}{r} = 2\pi$. Thus 2π radians $= 360°$ or π radians $= 180°$. Hence

$$1 \text{ radian} = \frac{180°}{\pi} = 57°\ 16' \text{ approximately}$$

and $1° = \frac{\pi}{180}$ radian $= 0.01746$ radian appproximately.

Note that the definition $t = \frac{s}{r}$ can be used to find one of s, t and r provided the other two are given.

Example 2.1

Find the length of the arc of a circle of radius 5 cm subtending an angle measuring 45°

Solution

$$t = \frac{\pi}{180} \times 45 = \frac{\pi}{4}. \text{ Hence } s = r \times t = \frac{5\pi}{4} \text{ cm}$$

Example 2.2

Suppose arcs of the same length in two circles subtend angles of 60° and 75° at the centre. Find the ratio of their radii.

Solution

Let r_1, r_2 denote the radii of the two circles. Now

$$60^\circ = \frac{\pi}{180} \times 60 = \frac{\pi}{3} \text{ and } 75^\circ = \frac{\pi}{180} \times 75 = \frac{5\pi}{12}$$

Let s be the length of the arcs. Then

$$s = \frac{\pi}{3}.r_1 = \frac{5\pi}{12}.r_2$$

Hence $r_1 : r_2 = 5 : 4$

Radian measure of some common angles are given in the following table:

Radians	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
Degrees	30°	45°	60°	90°	180°	270°	360°

Note that when an angle is expressed in radians, the word 'radians' is often omitted. Thus $\pi = 180^\circ$ is really a short form of writing π radians $= 180^\circ$.

EXERCISE 2.1

1. Find the radian measure corresponding to the following degree measures
   (a) 15° (b) $-22^\circ\ 30'$ (c) 340° (d) 420°
2. Find the degree measure corresponding to the following radian measures
   (a) $\frac{1}{4}$ (b) -2 (c) $\frac{7\pi}{3}$ (d) $\frac{5\pi}{6}$
3. Find the length of an arc of a circle of radius 5 cm subtending a central angle measuring 15°.
4. In a circle of diameter 40 cm, the length of a chord is 20 cm. Find the length of minor arc of the chord.
5. A wheel makes 180 revolutions in one minute. Through how many radians does it turn in one second?

6. Find in degrees the angle subtended at the centre of a circle of diameter 50 cm by an arc of 11 cm (use $\pi = \frac{22}{7}$).

7. Find the angle through which a pendulum swings if its length is 50 cm and the tip describes an arc of length (a) 10cm (b) 16 cm (c) 26 cm. (use $\pi = \frac{22}{7}$).

8. Find the angle between the minute hand of a clock and the hour hand when the time is 7.20.

2.3 Circular Functions or Trigonometric Functions

Let θ be the angle XOC as shown in Fig. 2.4 and let $P(x, y)$ be any point other than O on its terminal side OC. Let length of $OP = r$. We take always r to be > 0. We define the following functions known as circular or trigonometric functions. These are also called trigonometric ratios.

$$\text{sine } \theta = \sin\theta = \frac{y}{r}$$
$$\text{cosine } \theta = \cos\theta = \frac{x}{r}$$
$$\text{tangent } \theta = \tan\theta = \frac{y}{x} \quad (\theta \neq \frac{\pi}{2})$$
$$\text{cosecant } \theta = \operatorname{cosec}\theta = \frac{r}{y} \quad (\theta \neq 0)$$
$$\text{secant } \theta = \sec\theta = \frac{r}{x} \quad (\theta \neq \frac{\pi}{2})$$
$$\text{cotangent } \theta = \cot\theta = \frac{x}{y} (\theta \neq 0)$$

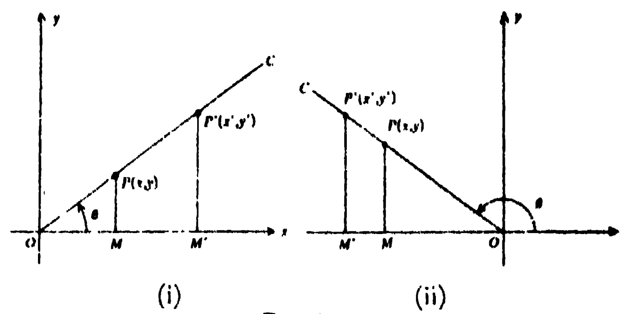


Fig 2.4

Note that these ratios may be positive or negative depending on x and/or y.

We observe that the above functions depend only on the value of the angle θ and not on the point P chosen on the terminal side of the angle θ. For example, if we take another point $P'(x', y')$ on OC with $OP' = r'$ then considering similar triangles we obtain

$$\frac{y}{r} = \frac{y'}{r'}, \frac{x}{r} = \frac{x'}{r'}, \frac{y}{x} = \frac{y'}{x'}$$

This is also true if the terminal side coincides with one of the axes with the only difference that if it coincides with x-axis, then cosec and cot are not defined while in case it coincides with y-axis, then sec and tan are not defined. From the definition and Fig. 2.5, it is clear that

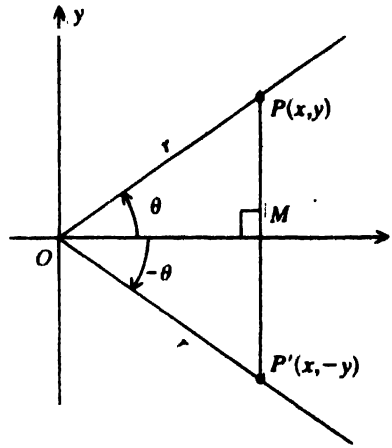


Fig.2.5

$$\sin\theta = \sin(\theta+2\pi) \quad \sin(-\theta) = -\sin\theta$$
$$\cos\theta = \cos(\theta+2\pi) \quad \cos(-\theta) = \cos\theta$$
$$\tan\theta = \tan(\theta+2\pi) \quad \tan(-\theta) = -\tan\theta$$

At this state, let us notice two important properties of trigonometric functions. Let us briefly introduce the notion of even and odd functions and that of periodic functions. A function f is said to be even if $f(x) = f(-x)$ for all x.

A simple example of an even function is a constant function.
Any polynomial function $p(x) = a_0 + a_1x^2 + a_2x^4 + \ldots + a_nx^{2n}$ (in which there are only even powers of x) is an even function.
We have already seen that cos $(-\theta) = \cos\theta$ for all θ. Thus cosine function is also an even function.
A function f is said to be *odd* if $f(-x) = -f(x)$ for all x.
It can be easily verified that the function $f(x) = x$, $f(x) = x^3$ are odd functions. In fact any polynomial function in which the coefficients of even powers of x are zero is an odd function. We have also seen that

$$\sin(-\theta) = -\sin\theta \text{ and}$$
$$\tan(-\theta) = -\tan\theta \text{ for all } \theta.$$

Thus sine and tangent functions are also odd.

The property of functions being even or odd is very useful in the study of such functions. It also helps in drawing graph of such functions as once we draw the graph for $x \geq 0$, we can complete the graph of f for all x easily.

The other important property of trigonometric functions which we want to observe now is that of periodicity. A function f is said to be *periodic* if there exists a real $T > 0$ such that $f(x+T) = f(x)$ for all x. We have already noted that

$$\sin(\theta+2\pi) = \sin\theta$$

$$\text{and } \cos(\theta+2\pi) = \cos\theta$$

Thus sine and cosine functions are examples of periodic functions. If a function f is periodic then the smallest $T > 0$, if it exists, such that

$$f(x+T) = f(x) \text{ for all } x$$

is called the period of the function. It can be easily seen that the period of the sine and cosine functions is 2π. We shall see later that the period of the tangent function is π. It is interesting to note that a constant function f is periodic as $f(x+T) = f(x)$ for all $T > 0$, however it does not have a period because there is no *smallest* $T > 0$ for which the relation holds.

The periodicity of a function is also very useful concept. In particular, it follows that the graph of such a function is completely known once we know it over an interval whose length is equal to the period of function. The periodicity of trigo nometric functions helps us to compute the value of these functions for large θ. For example,

$$\sin 785^\circ = \sin(2 \times 360^\circ + 65^\circ) = \sin 65^\circ$$

$$\text{and } \cos(-2070^\circ) = \cos(-2070^\circ + 6 \times 360^\circ) = \cos 90^\circ$$

Values of Trigonometric Functions for the Angles 0°, 30°, 45°, 60°, 90°.

From the definition of trigonometric functions it follows that the values of all trigonometric functions for given angle are known, once we find the sine and cosine of the angle.

It is clear from the definition that

$$\sin 0^\circ = 0,\ \cos 0^\circ = 1,\ \sin 90^\circ = 1, \text{ and } \cos 90^\circ = 0.$$

Let us calculate the values of these functions for 30°, 45°, 60°:

In Fig. 2.6, $PM = MP'$. The two triangles OPM and $OP'M$ are congruent. Hence, the triangle OPP' is an equilateral triangle. Therefore, if $PP' = 2a$, then $OP = 2a$, $PM = a$ and $OM = a\sqrt{3}$. Hence, $\sin 30^\circ = \frac{1}{2}$ and $\cos 30^\circ = \frac{\sqrt{3}}{2}$.

In Fig. 2.7, angles POM and OPM are equal. Hence, $OM = PM = a$(say). Then $OP = a\sqrt{2}$ and $\sin 45^\circ = \frac{1}{\sqrt{2}}$ and $\cos 45^\circ = \frac{1}{\sqrt{2}}$:

In Fig. 2.8, $OM = MM'$, the triangles OPM and $M'PM$ are congruent and the triangle OPM' is equilateral. Hence, if $OM = a$, then $OP = 2a$ and $PM = a\sqrt{3}$. Therefore, $\sin 60^\circ = \frac{\sqrt{3}}{2}$ and $\cos 60^\circ = \frac{1}{2}$.

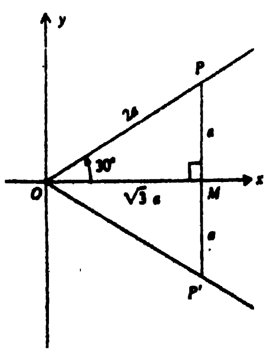


Fig 2.6

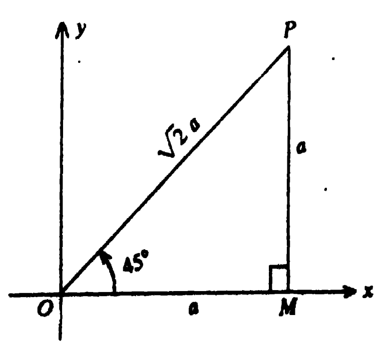


Fig 2.7

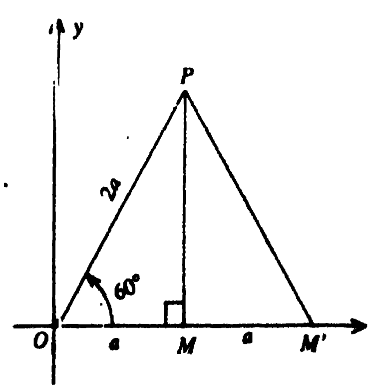


Fig 2.8

Thus we have the following table:

θ	$0°$	$30°$	$45°$	$60°$	$90°$
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined

Notational Convention

Since angles are measured either in radians or degrees, we adopt the convention that whenever we write $\cos\theta°$, we mean the cosine of the angle whose degree measure is $\theta°$ (and similarly for other ratios) and whenever we write $\cos\beta$, we mean the cosine of the angle whose measure in radians is β. If no such convention used, it should be clear from the context, which meaning is being used.

Signs of the Trigonometric Functions

The signs of the trigonometric functions depend on the quadrant in which the terminal arm of the angle lies. Thus $\sin\theta = \frac{y}{r}$ has the sign of y as r is always positive. Therefore $\sin\theta$ is taken as positive if the *angle* is in first or second quadrant, while it is negative for the *angles* in the third or fourth quadrant. Similarly $\cos\theta$ is positive in the first and fourth quadrants and negative in the remaining two quadrants. In fact we have the following table:

	I	II	III	IV
$\sin\theta$	+	+	−	−
$\cos\theta$	+	−	−	+
$\tan\theta$	+	−	+	−
$\operatorname{cosec}\theta$	+	+	−	−
$\sec\theta$	+	−	−	+
$\cot\theta$	+	−	+	−

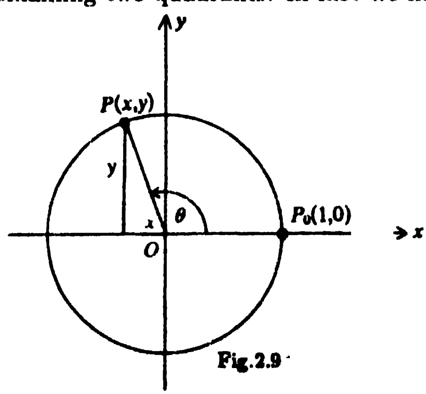


Fig.2.9

From the definition of trigonometric functions we observe the following useful facts. If P is the point on the circle with centre at origin O and radius 1 unit and if angle POX is θ then $\sin\theta = y$ and $\cos\theta = x$, where (x, y) are coordinates of P. It is also clear that $-1 \leq \sin\theta \leq 1$ and $-1 \leq \cos\theta \leq 1$ for all θ.

In the first quadrant as the angle increases from 0° to 90°, $\sin\theta$ increases from 0 to 1. In the second quadrant as θ increases from 90° to 180°, $\sin\theta$ decreases from 1 to 0. In the third quadrant as θ increases from 180° to 270°, $\sin\theta$ decreases from 0 to -1 and finally in the fourth quadrant $\sin\theta$ increases from -1 to 0 as θ increases from 270° to 360°. In fact we have the following table:

	I quadrant		*II quadrant*
sine	increases from 0 to 1	sine	decreases from 1 to 0
cosine	decreases from 1 to 0	cosine	decreases from 0 to -1
tangent	increases from 0 to ∞	tangent	increases from $-\infty$ to 0
cotangent	decreases from ∞ to 0	cotangent	decreases from 0 to $-\infty$
secant	increases from 1 to ∞	secant	increases from $-\infty$ to -1
cosecant	decreases from ∞ to 1	cosecant	increases from 1 to ∞

	III quadrant		*IV quadrant*
sine	decreases from 0 to -1	sine	increases from -1 to 0
cosine	increases from -1 to 0	cosine	increases from 0 to 1
tangent	increases from 0 to ∞	tangent	increases from $-\infty$ to 0
cotangent	decreases from ∞ to 0	cotangent	decreases from 0 to $-\infty$
secant	decreases from -1 to $-\infty$	secant	decreases from ∞ to 1
cosecant	increases from $-\infty$ to -1	cosecant	decreases from -1 to $-\infty$

Remark

In the above table we see the symbol ∞. Observe that ∞ is not a real number and is just a symbol. Statement like $\tan\theta$ increases from 0 to ∞ for $\theta \in (0, \frac{\pi}{2})$ simply means that $\tan\theta$ increases as θ increases in the interval $(0, \frac{\pi}{2})$ and assumes arbitrarily large positive values as θ increases to $\frac{\pi}{2}$. Similarly, to say that cosecant decreases from -1 to $-\infty$ in the fourth quadrant means that $\operatorname{cosec}\theta$ is a *decreasing function* for $\theta \in (\frac{3\pi}{2}, 2\pi)$ and assumes arbitrarily large negative values as θ approaches 2π.

2.4 Trigonometric Identities

An equation involving trigonometric functions which is true for all those angles for which the functions are defined is called a trigonometric identity. For example, $\sec\theta = \frac{1}{\cos\theta}$ is a trigonometric identity. It holds for all θ except those for which $\cos\theta = 0$.

An equation of the form $\sin\theta = \cos\theta$ is a trigonometric equation but not a trigonometric identity because it is not true for all θ. Trigonometric identities and solutions of trigonometric equations are very important and are useful in various problems of engineering and science.

Fundamental Identities

$$\sin\theta = \frac{1}{\operatorname{cosec}\theta} \qquad \tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\tan\theta = \frac{1}{\cot\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\sin^2\theta + \cos^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \operatorname{cosec}^2\theta$$

All of the above identities are very easy to prove and the proofs are left as an exercise. From these we also observe that given one of the six trigonometric functions, all others can be found numerically and their signs can be found by seeing in which quadrant the angle lies.

Example 2.3
Given $\cot\theta = \frac{12}{5}$, θ in quadrant III, find the values of the other five functions.

Solution

$$\tan\theta = \frac{5}{12}, \qquad \sec^2\theta = 1 + \tan^2\theta = 1 + \frac{25}{144} = \frac{169}{144}$$

Now in quadrant III, $\sin\theta$, $\cos\theta$, $\sec\theta$ and $\operatorname{cosec}\theta$ are all negative. Therefore

$$\sec\theta = -\frac{13}{12}, \qquad \cos\theta = -\frac{12}{13}, \qquad \sin\theta = \tan\theta\cos\theta = \frac{5}{12}\left(\frac{-12}{13}\right) = \frac{-5}{13},$$

$$\operatorname{cosec}\theta = \frac{-13}{5}$$

Example 2.4
Prove that $\sqrt{\frac{1-\sin A}{1+\sin A}} = \sec A - \tan A$

Solution

$$\text{L.H.S.}\ \frac{\sqrt{1-\sin A}}{\sqrt{1+\sin A}}\,\frac{\sqrt{1-\sin A}}{\sqrt{1-\sin A}} = \frac{1-\sin A}{\sqrt{1-\sin^2 A}} = \frac{1-\sin A}{\cos A} = \sec A - \tan A$$

$$= \text{R.H.S.}$$

Example 2.5
Prove that

$$\frac{\sin\theta}{1-\cos\theta}+\frac{\tan\theta}{1+\cos\theta}=\sec\theta\operatorname{cosec}\theta+\cot\theta$$

Solution

$$\begin{aligned}
\text{L.H.S.} &= \frac{\sin\theta}{1-\cos\theta}+\frac{\tan\theta}{1+\cos\theta}\\
&= \frac{\sin\theta+\sin\theta\cos\theta+\tan\theta-\tan\theta\cos\theta}{1-\cos^2\theta}\\
&= \frac{\sin\theta+\sin\theta\cos\theta+\tan\theta-\sin\theta}{\sin^2\theta}\\
&= \frac{\sin\theta\cos\theta+\tan\theta}{\sin^2\theta}\\
&= \cot\theta+\sec\theta\operatorname{cosec}\theta=\text{R.H.S.}
\end{aligned}$$

Example 2.6
Prove that

$$\frac{\tan A+\sec A-1}{\tan A-\sec A+1}=\frac{1+\sin A}{\cos A}$$

Solution

$$\begin{aligned}
\text{L.H.S.} &= \frac{\tan A+\sec A-1}{\tan A-\sec A+1}=\frac{\sin A+1-\cos A}{\sin A-1+\cos A}\\
&= \frac{(\sin A+1-\cos A)}{(\sin A+\cos A-1)}\times\frac{(\sin A+1+\cos A)}{(\sin A+1+\cos A)}\\
&= \frac{(\sin A+1)^2-\cos^2 A}{(\sin A+\cos A)^2-1}=\frac{1+2\sin A+\sin^2 A-\cos^2 A}{\sin^2 A+\cos^2 A+2\sin A\cos A-1}\\
&= \frac{2\sin A+2\sin^2 A}{2\sin A\cos A}\\
&= \frac{1+\sin A}{\cos A}=\text{R.H.S.}
\end{aligned}$$

EXERCISE 2.2

Find the values of the other five trigonometric functions in each of the following problems:

1. $\cos\theta=-\frac{1}{2}$, θ in quadrant II

2. $\sin\theta = \frac{3}{5}$, θ in quadrant I

3. $\tan\theta = \frac{3}{4}$, θ in quadrant III

Prove the following trigonometric identities:

4. $\tan^2\theta - \sin^2\theta = \tan^2\theta\sin^2\theta$

5. $\frac{\sin\theta}{1+\cos\theta} + \frac{1+\cos\theta}{\sin\theta} = 2\operatorname{cosec}\theta$

6. $\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \operatorname{cosec}\theta - \cot\theta$

7. $\frac{\tan\theta - \cot\theta}{\sin\theta\cos\theta} = \sec^2\theta - \operatorname{cosec}^2\theta$

8. $\frac{\operatorname{cosec}\theta}{\cot\theta + \tan\theta} = \cos\theta$

9. $\frac{\sec\theta - \tan\theta}{\sec\theta + \tan\theta} = 1 - 2\sec\theta\tan\theta + 2\tan^2\theta$

10. $\frac{1-\cos\theta}{\sin\theta} = \frac{\sin\theta}{1+\cos\theta}$

11. $\sec^4\theta - \sec^2\theta = \tan^4\theta + \tan^2\theta$

12. $\sin^8\theta - \cos^8\theta = (\sin^2\theta - \cos^2\theta)(1 - 2\sin^2\theta\cos^2\theta)$

13. $2\sec^2\theta - \sec^4\theta - 2\operatorname{cosec}^2\theta + \operatorname{cosec}^4\theta = \cot^4\theta - \tan^4\theta$

2.5 Cosine of the Difference of Two Angles

We begin by establishing a formula for the cosine of the difference of two angles in terms of sines and cosines of the individual angles. We shall see that this helps us in proving several other important identities.

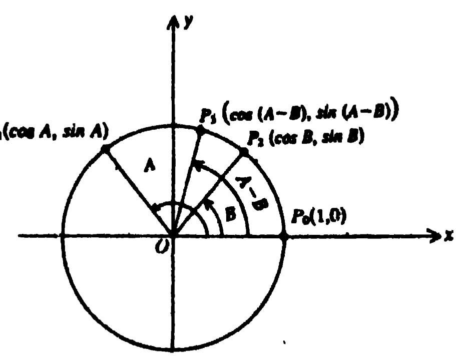


Fig 2.10 (i)

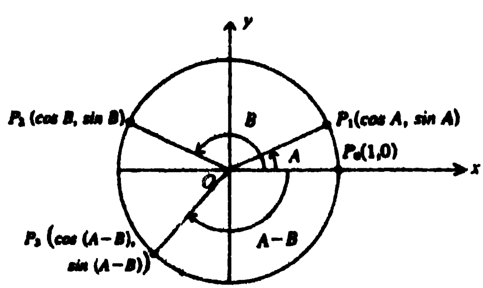


Fig 2.10 (ii)

Recall that the terminal side of any angle cuts the circle with centre at O and unit radius at a point whose coordinates are respectively the cosine and sine of the angle. In Fig. 2.10 OP_1 and OP_2 are the terminal sides of the angles A and B respectively and OP_3 has been drawn to be the terminal side of the angle $A-B$. It is now clear that the chords P_0P_3, and P_1P_2 subtend the central angles of same size and hence are equal in length. Therefore, we obtain $[\cos(A-B)-1]^2+\sin^2(A-B) = (\cos B-\cos A)^2+(\sin B-\sin A)^2$. This on simplification yields

$$\cos(A-B) = \cos A\cos B + \sin A\sin B.$$

The method of proof of the above formula applied to all values of A and B. Hence, the formula holds for all angles A and B, positive, zero or negative.

Example 2.7

Find the values of $\cos 15°$ and $\cos 75°$.

Solution

$$\begin{aligned}\cos 15° = \cos(45° - 30°) &= \cos 45°\cos 30° + \sin 45°\sin 30°\\ &= \frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}.\frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}\end{aligned}$$

$$\begin{aligned}\text{Similarly,}\quad \cos 75° &= \cos(45° - (-30°))\\ &= \cos 45°\cos 30° + \sin 45°\sin(-30°)\\ &= \cos 45°\cos 30° - \sin 45°\sin 30°\\ &= \frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}.\frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}\end{aligned}$$

Ratios of Complementary Angle

$$\begin{aligned}\cos(90° - \theta) &= \cos 90°\cos\theta + \sin 90°\sin\theta\\ &= \sin\theta\\ \sin(90° - \theta) &= \cos(90° - (90° - \theta)) = \cos\theta\end{aligned}$$

Hence, we have

$$\begin{aligned}\sin(90° - \theta) &= \cos\theta & \quad \operatorname{cosec}(90° - \theta) &= \sec\theta\\ \cos(90° - \theta) &= \sin\theta & \sec(90° - \theta) &= \operatorname{cosec}\theta\\ \tan(90° - \theta) &= \cot\theta & \cot(90° - \theta) &= \tan\theta\end{aligned}$$

From the above, replacing θ by $-\theta$ and recalling the formulas on section no. 2.3, we obtain

$$\begin{array}{lllllll}
\sin(90^\circ+\theta) & = & \cos\theta & & \operatorname{cosec}(90^\circ+\theta) & = & \sec\theta \\
\cos(90^\circ+\theta) & = & -\sin\theta & & \sec(90^\circ+\theta) & = & -\operatorname{cosec}\theta \\
\tan(90^\circ+\theta) & = & -\cot\theta & & \cot(90^\circ+\theta) & = & -\tan\theta
\end{array}$$

Formulas for Functions of Sum and Difference of Two Angles

$$\begin{aligned}
\cos(A+B) &= \cos(A-(-B)) \\
&= \cos A\cos(-B)+\sin A\sin(-B) \\
&= \cos A\cos B-\sin A\sin B \\
\sin(A-B) &= \cos(90^\circ-(A-B)) \\
&= \cos((90^\circ-A)+B) \\
&= \cos(90^\circ-A)\cos B-\sin(90^\circ-A)\sin B \\
&= \sin A\cos B-\cos A\sin B \\
\sin(A+B) &= \sin(A-(-B)) \\
&= \sin A\cos(-B)-\cos A\sin(-B) \\
&= \sin A\cos B+\cos A\sin B \\
\tan(A+B) &= \frac{\sin(A+B)}{\cos(A+B)} \\
&= \frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B} \\
&= \frac{\tan A+\tan B}{1-\tan A\tan B} \quad \text{(divide numerator and denominator by } \cos A\cos B\text{)}
\end{aligned}$$

Similar computations yield

$$\begin{aligned}
\cot(A+B) &= \frac{\cot A\cot B-1}{\cot B+\cot A} \\
\tan(A-B) &= \frac{\tan A-\tan B}{1+\tan A\tan B} \\
\cot(A-B) &= \frac{\cot A\cot B+1}{\cot B-\cot A}
\end{aligned}$$

Using above formulas, we get

$$\begin{array}{lllllll}
\sin(180^\circ-\theta) & = & \sin\theta & & \operatorname{cosec}(180^\circ-\theta) & = & \operatorname{cosec}\theta \\
\cos(180^\circ-\theta) & = & -\cos\theta & & \sec(180^\circ-\theta) & = & -\sec\theta \\
\tan(180^\circ-\theta) & = & -\tan\theta & & \cot(180^\circ-\theta) & = & -\cot\theta \\
\sin(180^\circ+\theta) & = & -\sin\theta & & \operatorname{cosec}(180^\circ+\theta) & = & -\operatorname{cosec}\theta \\
\cos(180^\circ+\theta) & = & -\cos\theta & & \sec(180^\circ+\theta) & = & -\sec\theta \\
\tan(180^\circ+\theta) & = & \tan\theta & & \cot(180^\circ+\theta) & = & \cot\theta
\end{array}$$

From this we conclude that the period of the tangent function is π. All these can be directly deduced from the definitions given in section 2.3.
Also,

$$\begin{aligned}
2\sin A\cos B &= \sin(A+B)+\sin(A-B)\\
2\cos A\sin B &= \sin(A+B)-\sin(A-B)\\
2\cos A\cos B &= \cos(A+B)+\cos(A-B)\\
2\sin A\sin B &= \cos(A-B)-\cos(A+B)
\end{aligned}$$

These are called product formulas.
We also have the following sum formulas.

$$\begin{aligned}
\sin A+\sin B &= \sin\left(\frac{A+B}{2}+\frac{A-B}{2}\right)+\sin\left(\frac{A+B}{2}-\frac{A-B}{2}\right)\\
&= 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}
\end{aligned}$$

Similarly,

$$\begin{aligned}
\sin A-\sin B &= 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}\\
\cos A+\cos B &= 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}\\
\cos A-\cos B &= -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}
\end{aligned}$$

Values of Functions at $2A$, $\frac{1}{2}A$ and $3A$

$$\begin{aligned}
\sin 2A=\sin(A+A) &= 2\sin A\cos A\\
\cos 2A=\cos(A+A) &= \cos^2 A-\sin^2 A\\
&= 2\cos^2 A-1\\
&= 1-2\sin^2 A\\
\tan 2A &= \tan(A+A)\\
&= \frac{2\tan A}{1-\tan^2 A}
\end{aligned}$$

From the above $2\cos^2 A = 1+\cos 2A$ and
$2\sin^2 A = 1-\cos 2A$

Hence, replacing A by $\frac{A}{2}$, in the above formula, we get

$$\sin\frac{1}{2}A = \pm\sqrt{\frac{1-\cos A}{2}}$$

$$\cos\frac{1}{2}A = \pm\sqrt{\frac{1+\cos A}{2}}$$

$$\tan\frac{1}{2}A = \pm\sqrt{\frac{1-\cos A}{1+\cos A}}$$

$$\begin{aligned}
\text{Also } \tan\frac{1}{2}A &= \frac{1-\cos A}{\sin A} \quad \frac{\sin A}{1+\cos A} \\
\text{Again } \sin 3A &= \sin(A+2A) \\
&= \sin A\cos 2A + \cos A\sin 2A \\
&= \sin A(1-2\sin^2 A) + 2\sin A(1-\sin^2 A) \\
&= 3\sin A - 4\sin^3 A.
\end{aligned}$$

Similarly we can show that

$$\cos 3A = 4\cos^3 A - 3\cos A$$

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1-3\tan^2 A}.$$ The student is advised to prove these.

Values of sine and cosine for Some Special Angles

We have already found the values of $\cos 15^\circ$ and $\cos 75^\circ$. Recall that once we know the values of $\cos 15^\circ$ and $\cos 75^\circ$, it is easy to find the values of $\sin 15^\circ$ and $\sin 75^\circ$. Thus

$$\sin 15^\circ = \sin(90^\circ - 75^\circ) = \cos 75^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

$$\text{and } \sin 75^\circ = \cos 15^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

Trigonometric Functions of an Angle of 18°

Let $\theta = 18^\circ$. Then $2\theta = 90^\circ - 3\theta$

Therefore

$$\begin{aligned}
\sin 2\theta &= \cos 3\theta \\
\text{or } 2\sin\theta\cos\theta &= 4\cos^3\theta - 3\cos\theta.
\end{aligned}$$

Since $\cos\theta \neq 0$, we get

$2\sin\theta = 4\cos^2\theta - 3 = 1 - 4\sin^2\theta$

or $4\sin^2\theta + 2\sin\theta - 1 = 0$

Hence $\sin\theta = \frac{-2 \pm \sqrt{4+16}}{2\times 4} = \frac{-1\pm\sqrt{5}}{4}$

Since $\theta = 18°, \sin\theta > 0$. Therefore $\sin 18° = \frac{\sqrt{5}-1}{4}$

Also $\cos 18° = \sqrt{1-\sin^2 18°} = \sqrt{1-\frac{6-2\sqrt{5}}{16}} = \sqrt{\frac{10+2\sqrt{5}}{16}}$

Hence, $\cos 18° = \frac{\sqrt{10+2\sqrt{5}}}{4}$.

Now we can easily find $\cos 36°$, $\sin 36°$.

$$\begin{aligned}\cos 36° &= 1-2\sin^2 18° = 1-\frac{2(6-2\sqrt{5})}{16}\\ &= \frac{2+2\sqrt{5}}{8} = \frac{\sqrt{5}+1}{4}.\\ \text{Hence, } \cos 36° &= \frac{\sqrt{5}+1}{4}.\end{aligned}$$

$$\begin{aligned}\text{Also } \sin 36° &= \sqrt{1-\cos^2 36°} = \sqrt{1-\frac{6+2\sqrt{5}}{16}}\\ &= \frac{\sqrt{10-2\sqrt{5}}}{4}\end{aligned}$$

Example 2.8

Prove that

$$\sin 75° - \sin 15° = \cos 105° + \cos 15°$$

Solution

$\cos 15° = \sin(90° - 15°) = \sin 75°$

$\cos 105° = \cos(90° + 15°) = -\sin 15°$

Therefore, $\cos 105° + \cos 15° = \sin 75° - \sin 15°$

Example 2.9

Prove that

$$\frac{\sin(x-y)}{\sin(x+y)} = \frac{\tan x - \tan y}{\tan x + \tan y}$$

Solution

L.H.S. $= \frac{\sin x\cos y - \cos x\sin y}{\sin x\cos y + \cos x\sin y}$

Dividing the numerator and denominator by $\cos x\cos y$, we get that

$$\text{L.H.S.} = \frac{\tan x - \tan y}{\tan x + \tan y} = \text{R.H.S.}$$

Example 2.10
Prove that

$$\frac{\sin 5A - \sin 3A}{\cos 3A + \cos 5A} = \tan A$$

Solution

$$\text{L.H.S.} = \frac{2\cos 4A \ \sin A}{2\cos 4A \ \cos A} = \tan A = \text{R.H.S.}$$

Example 2.11
Prove that

$$\cos 2\theta \cos\frac{\theta}{2} - \cos 3\theta \cos\frac{9\theta}{2} = \sin 5\theta \sin\frac{5\theta}{2}$$

Solution

$$\begin{aligned}
\text{L.H.S.} &= \frac{1}{2}\left[2\cos 2\theta \cos\frac{\theta}{2} - 2\cos 3\theta \cos\frac{9\theta}{2}\right] \\
&= \frac{1}{2}\left[\cos\left(2\theta + \frac{\theta}{2}\right) + \cos\left(2\theta - \frac{\theta}{2}\right) - \cos\left(3\theta + \frac{9\theta}{2}\right) - \cos(3\theta - \frac{9\theta}{}\right. \\
&= \frac{1}{2}\left[\cos\frac{5\theta}{2} + \cos\frac{3\theta}{2} - \cos\frac{15\theta}{2} - \cos\frac{3\theta}{2}\right] \\
&= \frac{1}{2}\left[\cos\frac{5\theta}{2} - \cos\frac{15\theta}{2}\right] \\
&= \sin\frac{5\theta + 15\theta}{4} \sin\frac{15\theta - 5\theta}{4} \\
&= \sin 5\theta \sin\frac{5\theta}{2} = \text{R.H.S.}
\end{aligned}$$

Example 2.12
Prove that

$$\frac{\sin 2A}{1 + \cos 2A} = \tan A$$

Solution

$$\text{L.H.S.} = \frac{2\sin A\cos A}{2\cos^2 A} = \tan A = \text{R.H.S}$$

Example 2.13
Prove that

$$\sin 4A = 4\sin A\cos^3 A - 4\cos A\sin^3 A$$

Solution

$$\begin{aligned}\text{L.H.S.} &= 2\sin 2A\cos 2A\\ &= 4\sin A\cos A(\cos^2 A - \sin^2 A)\\ &= 4\sin A\cos^3 A - 4\cos A\sin^3 A\\ &= \text{R.H.S.}\end{aligned}$$

Example 2.14
Find the value of $\tan 22°30'$

Solution

$$\text{Let } \theta = 45° \quad \text{Then } \frac{\theta}{2} = 22°30'$$

$$\begin{aligned}\text{Now } \tan\frac{\theta}{2} &= \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}\\ &= \frac{\sin\theta}{1+\cos\theta} = \frac{\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}} = \frac{1}{\sqrt{2}+1}\\ &= \frac{1}{\sqrt{2}+1}\cdot\frac{\sqrt{2}-1}{\sqrt{2}-1} = \sqrt{2}-1\end{aligned}$$

Hence, $\tan 22°30' = \sqrt{2}-1$

EXERCISE 2.3

1. If $\sin\alpha = \frac{15}{17}$ and $\cos\beta = \frac{12}{13}$, find the values of $\sin(\alpha+\beta), \cos(\alpha-\beta)$ and $\tan(\alpha+\beta)$.
2. Prove that

$$\cos(45° - A)\cos(45° - B) - \sin(45° - A)\sin(45° - B) = \sin(A + B)$$

3. Show that

$$\sin 105° + \cos 105° = \cos 45°$$

4. Prove that

$$\sin(n+1)A\sin(n+2)A+\cos(n+1)A\cos(n+2)A=\cos A$$

5. Prove that

$$\frac{\tan(45^\circ+x)}{\tan(45^\circ-x)}=\left[\frac{1+\tan x}{1-\tan x}\right]^2$$

6. $\dfrac{\sin A+\sin 3A}{\cos A+\cos 3A}=\tan 2A$

7. $\dfrac{\tan 5\theta+\tan 3\theta}{\tan 5\theta-\tan 3\theta}=4\cos 2\theta\cos 4\theta$

8. $\dfrac{\sin A+\sin B}{\cos A+\cos B}=\tan\left(\dfrac{A+B}{2}\right)$

9. $\cos 4x=1-8\sin^2 x\cos^2 x$

10. $\tan 4\theta=\dfrac{4\tan\theta(1-\tan^2\theta)}{1-6\tan^2\theta+\tan^4\theta}$

11. $(\sin 3A+\sin A)\sin A+(\cos 3A-\cos A)\cos A=0$

12. $2\cos\frac{\pi}{13}\cos\frac{9\pi}{13}+\cos\frac{3\pi}{13}+\cos\frac{5\pi}{13}=0$

Find sine, cosine and tangent of $\frac{x}{2}$ if $0\le x\le 2\pi$ in the following problems:

13. $\tan x=-\frac{4}{3}$, x in quadrant II

14. $\cos x=-\frac{1}{3}$, x in quadrant III

15. $\sin x=\frac{1}{4}$, x in quadrant II

16. Find $\sin 7\frac{1}{2}^\circ$, $\cos 7\frac{1}{2}^\circ$ and $\tan 11\frac{1}{4}^\circ$.

Prove that

17. $\sin^2 72^\circ-\sin^2 60^\circ=\dfrac{\sqrt{5}-1}{8}$

18. $\sin\frac{\pi}{5}\sin\frac{2\pi}{5}\sin\frac{3\pi}{5}\sin\frac{4\pi}{5}=\frac{5}{16}$

Prove that

19. $\sin\frac{\pi}{10}+\sin\frac{13\pi}{10}=-\frac{1}{2}$

20. $(\cos\alpha+\cos\beta)^2+(\sin\alpha+\sin\beta)^2=4\cos^2\dfrac{\alpha-\beta}{2}$

21. $\cos^2 A+\cos^2(A+120^\circ)+\cos^2(A-120^\circ)=\frac{3}{2}$

22. $\dfrac{\cos 4x+\cos 3x+\cos 2x}{\sin 4x+\sin 3x+\sin 2x}=\cot 3x$

23. $\sin 3A+\sin 2A-\sin A=4\sin A\cos\frac{A}{2}\cos\frac{3A}{2}$

24. $\cos 6A=32\cos^6 A-48\cos^4 A+18\cos^2 A-1$

25. $\tan 3A\tan 2A\tan A=\tan 3A-\tan 2A-\tan A$

2.6 Tables of Trigonometric Functions

In order to deal with many problems in trigonometry, it is necessary to find the values of trigonometric functions for various angles. Trigonometric functions of any angle can be computed to any *desired degree of accuracy*. Tables are available which give approximate values of the sine and tangent of angles from 0° to 90°. The values of $\cos\theta$ and $\cot\theta$ can also be easily read out by using formulas like $\sin(90° - \theta) = \cos\theta$, $\tan(90° - \theta) = \cot\theta$; $\sec\theta$ and $\operatorname{cosec}\theta$ can be found out by noticing that $\sec\theta = \frac{1}{\cos\theta}$, $\operatorname{cosec}\theta = \frac{1}{\sin\theta}$. There are tables which give the values of six trigonometric functions of angles from 0° to 90° For angles larger than 90° we use various formulas to reduce the value of trigonometric function to the numerical value of some trigonometric function of an angle between 0° and 90°. For example $\sin 124° = \sin(90° + 34°) = \cos 34°$. If sine of some angle is not given in the table we can use linear interpolation to find its value. We illustrate these by some examples.

Example 2.15
Find $\cot 131°20'$.

Solution
First we observe

$$\cot 131°20' = -\tan 41°20'$$

We are using $\cot(90° + \theta) = -\tan\theta$
From the table, we see that $\tan 41°20' = .8796$.
Hence $\cot 131°20' = -.8796$

Example 2.16
Find the angle θ if $\sin\theta = .7071$

Solution
In the table of sines, we see that $\sin 45° = .7071$
Hence, $\theta = 45°$

Example 2.17
Find the value of $\sin 23°26'$.

Solution
From the table we see that

$$\sin 23°20' = .3961$$

and $\sin 23°30' = .3987$ $\quad\therefore$ difference $= 0.0026$

difference on $10'$ is .0026.

$$\text{Hence, the difference for } 6' = \frac{6}{10} \times .0026 = .00156$$
$$= .0016 \text{ (approx.)}$$
$$\text{Hence, } \sin 23°26' = .3961 + .0016$$
$$= .3977$$

Example 2.18

Find θ if $\cot\theta = .5750$.

Solution

$\tan(90° - \theta) = \cot\theta = .5750$

From table we see $.5735 = \tan 29°50'$

and $.5774 = \tan 30°$

difference $= .0039$ Also $.5750 - .5735 = .0015$

For $.0039$ difference angle is $10'$

For $.0015$ difference angle is $\frac{10 \times 15}{39} = 4'$ approx.

Hence $90° - \theta = 29°54'$. Therefore $\theta = 60°6'$.

EXERCISE 2.4

1. Find the following:
   (i) $\cos 20°10'$ (ii) $\sin 48°$ (iii) $\tan 54°30'$ (iv) $\cot 33°40'$
2. Find the angle $\theta, 0° \leq \theta \leq 90°$, if
   (i) $\sin\theta = 0.5373$ (ii) $\cos\theta = .0087$
   (iii) $\tan\theta = 34.37$ (iv) $\cot\theta = 3.018$
3. Find the following:
   (i) $\sin 34°22'$ (ii) $\cos 64°34'$
   (iii) $\tan 42°6'$ (iv) $\cot 47°26'$
4. Find θ where
   (i) $\sin\theta = .5240$ (ii) $\cos\theta \pm .0424$
   (iii) $\cot\theta = 1.246$ (iv) $\tan\theta = .1362$

2.7 Graphs of Trigonometric Functions

We have already seen that all trigonometric functions are periodic. For example, the period of the sine and cosine functions is 2π, while that of the tangent functions is π. Often, we have to deal with functions of the form $\sin ax$, $c\cos(ax + b)$ and so on. These

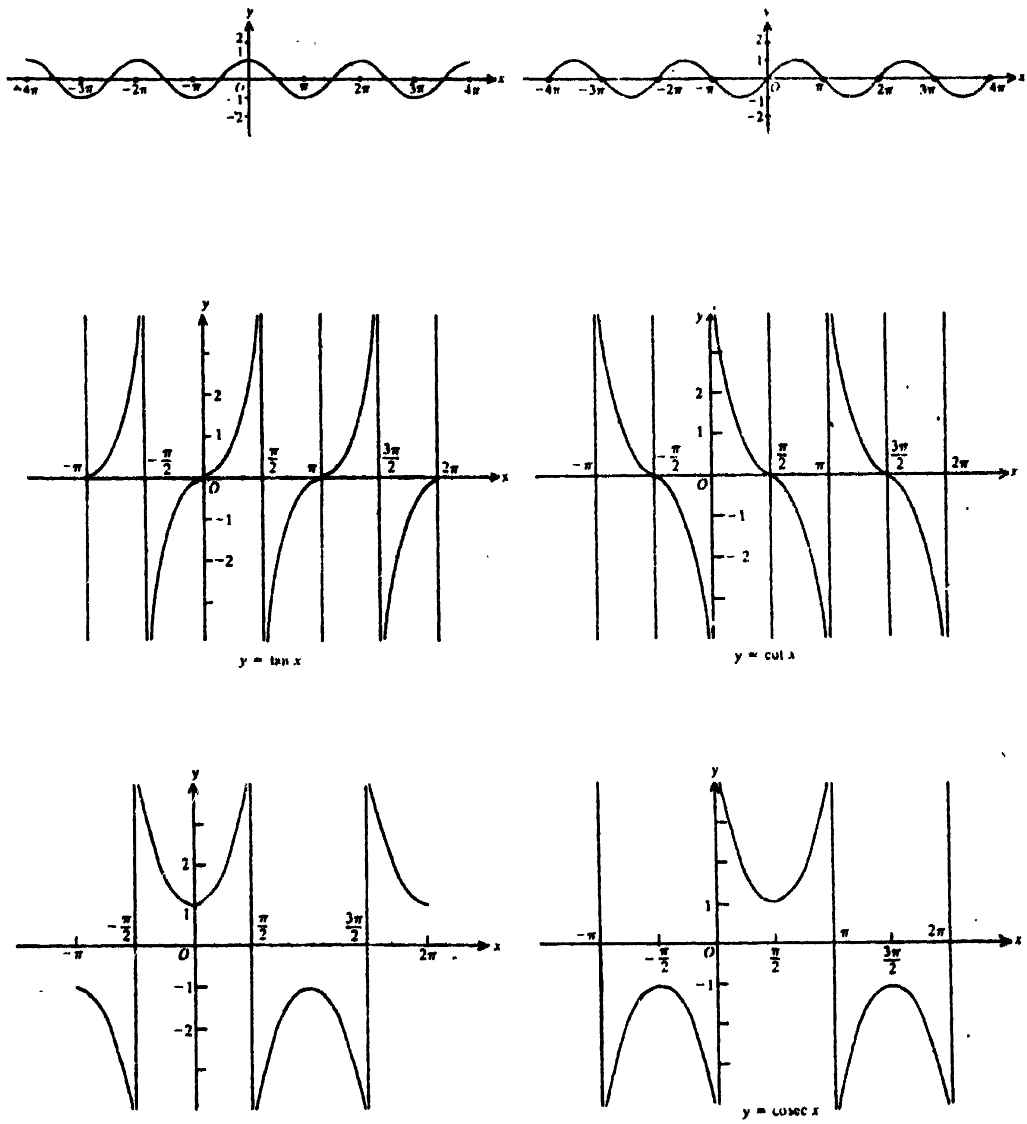


Fig. 2.11

functions are also periodic with period $\frac{2\pi}{a}$ as can be easily verified. For example, the period of $5\sin(3x+4)$ is $\frac{2\pi}{3}$.

The graph of any periodic function with period T need to be sketched only in an interval of length T, as once it is drawn in one such interval, it can be easily drawn completely by repeating it over other intervals of length T.

The graph of $y = \sin x$.

We have the following table

x	0°	30°	45°	60°	90°
$\sin x$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}} = .707$	$\frac{\sqrt{3}}{2} = .866$	1

We also have $\sin(\pi - t) = \sin t$ and $\sin(\pi + t) = -\sin t$. To draw the graph of $\sin x$ in $[0,2\pi]$, we first draw it in $[0, \frac{\pi}{2}]$ using the above table, recalling that in this interval the function is increasing. Then in $[\frac{\pi}{2}, \pi]$, we draw it using $\sin(\pi - t) = \sin t$. Finally, in $[\frac{\pi}{2}, \pi]$, we draw it using the fact $\sin(\pi + t) = -\sin t$. The graph is now sketched in Fig. 2.11. The graphs of other trigonometric functions $y = \cos x$, $y = \tan x$, $y = \cot x$, $y = \sec x$, $y = \operatorname{cosec} x$ are also given in Fig. 2.11.

Note: Observe that $\sin\theta = 0$ for $\theta = n\pi$ and $\cos\theta = 0$ for $\theta = (2n+1)\frac{\pi}{2}$, where n is any integer.

Example 2.19

Graph $f(x) = 3\cos 2x$

Let $y = 3\cos 2x$ $\therefore$ $-3 \le y \le 3$

Period of $f = \frac{2\pi}{2} = \pi$.

The graph of the curve in the interval $[0, \pi]$ is sketched in Fig. 2.12.

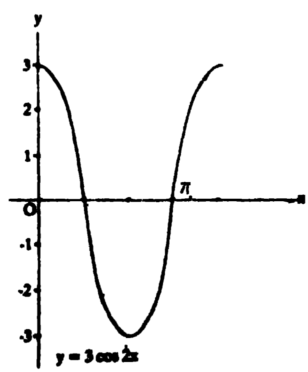


Fig 2.12

Example 2.20

Graph $f(x) = 3\sin(2x - 1)$

Let $y = 3\sin(2x - 1)$

Period of $f = \dfrac{2\pi}{2} = \pi$

range $-3 \leq y \leq 3$.

Suppose we wish to write

$y = 3\sin(2x - 1) = 3\sin 2t$

or $2x - 1 = 2t$

or $t = x - \dfrac{1}{2}$

or $x = t + \dfrac{1}{2}$

Thus if we draw the graph of $3\sin 2t$ and 'shift' it by $\frac{1}{2}$ to the right, we get the required graph. The graph is drawn in Fig. 2.13.

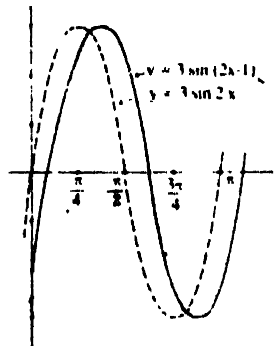


Fig 2.13

Example 2.21

Sketch the graph of $y = \sin x$ and $y = \sin 2x$ on the same axes.

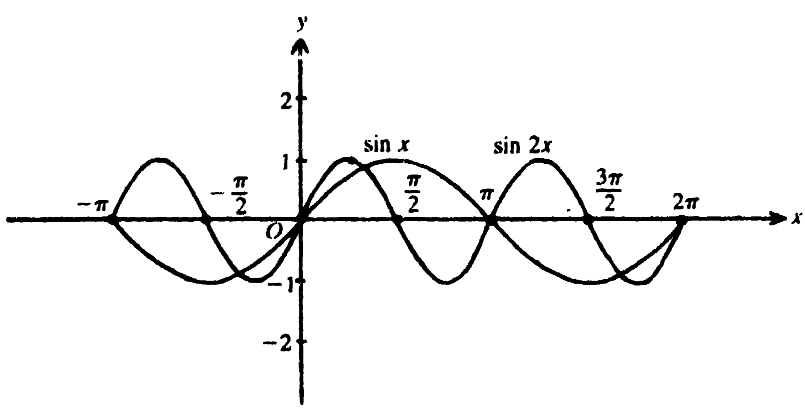


Fig 2.14

Solution

$$\begin{aligned}
y &= \sin x \\
\text{Range} &= \{y : -1 \le y \le 1\} \\
\text{Period} &= 2\pi \\
y &= \sin 2x \\
\text{Range} &= \{y : -1 \le y \le 1\} \\
\text{Period} &= \pi
\end{aligned}$$

The graphs are drawn in Fig. 2.14.

EXERCISE 2.5

Sketch the following graphs:

1. $y = \tan 3x$. 2. $y = 3\sin 2x$ 3. $y = \cos(x - \frac{\pi}{4})$

4. $y = 3\sin(3x + 1)$ 5. $y = x\sin^2 x$ 6. $y = \cos^2 x$

Sketch the graph of the following pair of equations on the same axes:

7. $y = \cos x, y = \cos \frac{1}{2}x$ 8. $y = \sin x, y = \sin(x + 45^\circ)$

9. $y = \tan x, y = \tan(x - 45^\circ)$ 10. $y = \cos 2x, y = \cos(2x - \pi)$

2.8 Conditional Identities Involving the Angles of a Triangle

When A, B, C are angles of a triangle, many identities hold between their trigonometric functions. We illustrate the method of proof by some examples.

Example 2.22
If $A + B + C = \pi$, prove that

$$\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

Solution

$$\begin{aligned}
\text{L.H.S.} &= 2\sin\frac{A+B}{2}\cos\frac{A-B}{2} + \sin C \\
&= 2\sin\left(90^\circ - \frac{C}{2}\right)\cos\frac{A-B}{2} + 2\sin\frac{C}{2}\cos\frac{C}{2} \\
&= 2\cos\frac{C}{2}\left(\cos\frac{A-B}{2} + \cos\left(90^\circ - \frac{C}{2}\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&= 2\cos\frac{C}{2}\left(\cos\frac{A-B}{2}+\cos\frac{A+B}{2}\right)\\
&= 2\cos\frac{C}{2}\left(2\cos\frac{A}{2}\cos\frac{B}{2}\right)\\
&= 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = \text{R.H.S.}
\end{aligned}
$$

Example 2.23
If $A+B+C=\pi$, show that

$$\cos 2A+\cos 2B+\cos 2C = -1-4\cos A\cos B\cos C$$

Solution

$$
\begin{aligned}
\text{L.H.S.} &= 2\cos(A+B)\cos(A-B)+\cos 2C\\
&= 2\cos(\pi-C)\cos(A-B)+\cos 2C\\
&= -2\cos C\cos(A-B)+2\cos^2 C-1\\
&= -1+2\cos C\{\cos C-\cos(A-B)\}\\
&= -1+2\cos C\left[\cos\{\pi-(A+B)\}-\cos(A-B)\right]\\
&= -1-2\cos C\{\cos(A+B)+\cos(A-B)\}\\
&= -1-2\cos C\{2\cos A\cos B\}\\
&= -1-4\cos A\cos B\cos C = \text{R.H.S.}
\end{aligned}
$$

Example 2.24
If $A+B+C=\pi$, show that

$$\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1$$

Solution

$$\tan\frac{A+B}{2} = \tan(90^\circ-\frac{C}{2}) = \cot\frac{C}{2}$$

$$\text{Also } \tan\frac{A+B}{2} = \frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{1-\tan\frac{A}{2}\tan\frac{B}{2}}$$

$$\text{Hence, } \frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{1-\tan\frac{A}{2}\tan\frac{B}{2}} = \frac{1}{\tan\frac{C}{2}}$$

or $$\tan\frac{A}{2}\tan\frac{C}{2}+\tan\frac{B}{2}\tan\frac{C}{2}=1-\tan\frac{A}{2}\tan\frac{B}{2}$$

or $$\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1,$$

which is what we wanted to prove.

EXERCISE 2.6

If $A+B+C=\pi$, show that

1. $\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C$
2. $\cos 2A+\cos 2B-\cos 2C=1-4\sin A\sin B\cos C$
3. $\cos A+\cos B+\cos C=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$
4. $\cos^2 A+\cos^2 B-\cos^2 C=1-2\sin A\sin B\cos C$
5. $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}-\sin^2\frac{C}{2}=1-2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$
6. $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$
7. $\cot B\cot C+\cot C\cot A+\cot A\cot B=1$
8. $\tan A+\tan B+\tan C=\tan A\tan B\tan C$
9. $\dfrac{\sin 2A+\sin 2B+\sin 2C}{\sin A+\sin B+\sin C}=8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$
10. $\tan 2A+\tan 2B+\tan 2C=\tan 2A\tan 2B\tan 2C$

2.9 Trigonometric Equations

In solving trigonometric equations, we come across situations of the type $\sin\theta=\sin\alpha$ or $\cos\theta=\cos\alpha$ or $\tan\theta=\tan\alpha$. Before illustrating the technique of solving trigonometric equations by some examples, we find the general value of all angles having a given sine, cosine or tangent.

1. Suppose $\sin\theta=\sin\alpha$
   Then $\sin\theta-\sin\alpha=0$
   or $2\cos\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2}=0$
   Hence, either $\cos\frac{\theta+\alpha}{2}=0$ or $\sin\frac{\theta-\alpha}{2}=0$.
   Therefore, either

$$\frac{\theta+\alpha}{2}=\text{ any odd multiple of }\pi/2$$

or $$\frac{\theta-\alpha}{2}=\text{ any multiple of }\pi\text{ (See section 2.7)}$$

Thus either

$$\theta = -\alpha + \text{any odd multiple of } \pi$$

or $$\theta = \alpha + \text{any even multiple of } \pi$$

Thus the general value of θ such that $\sin\theta = \sin\alpha$ is given by

$$\theta = n\pi + (-1)^n \alpha, \text{ where } n \text{ is an integer.}$$

2. Suppose $\cos\theta = \cos\alpha$
   Then $\cos\theta - \cos\alpha = 0$
   Hence, $-2\sin\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2} = 0.$
   Therefore, either $\frac{\theta+\alpha}{2} =$ any multiple of π.
   or $$\frac{\theta-\alpha}{2} = \text{any multiple of } \pi.$$
   Hence, the general solution for θ such that $\cos\theta = \cos\alpha$ is given by
   $$\theta = 2n\pi \pm \alpha, \text{ where } n \text{ is an integer.}$$

3. Suppose $\tan\theta = \tan\alpha$
   Then $\frac{\sin\theta}{\cos\theta} = \frac{\sin\alpha}{\cos\alpha}$
   or $\sin\theta\cos\alpha - \sin\alpha\cos\theta = 0$
   or $\sin(\theta - \alpha) = 0$
   or $\theta - \alpha = n\pi$.
   Hence the general solution for θ satisfying $\tan\theta = \tan\alpha$ is
   $\theta = n\pi + \alpha$, where n is an integer.

Example 2.25

Solve the equation

$$\sin\theta + \sin 3\theta + \sin 5\theta = 0$$

Solution

Given $\sin\theta + \sin 3\theta + \sin 5\theta = 0$

or $$(\sin\theta + \sin 5\theta) + \sin 3\theta = 0$$

or $$2\sin\frac{5\theta+\theta}{2}\cos\frac{5\theta-\theta}{2} + \sin 3\theta = 0$$

or $$2\sin 3\theta\cos 2\theta + \sin 3\theta = 0$$

or $$\sin 3\theta(2\cos 2\theta + 1) = 0$$

or $$\sin 3\theta = 0 \qquad \text{or, } \cos 2\theta = -\frac{1}{2}$$

When $\sin 3\theta = 0$, then $3\theta = n\pi$ or $\theta = \frac{n\pi}{3}$.

When $\cos 2\theta = -\frac{1}{2}$, then $2\theta = 2n\pi \pm \frac{2\pi}{3}$ or $\theta = n\pi \pm \frac{\pi}{3}$.

This yields $\theta = (3n+1)\frac{\pi}{3}$ or $\theta = (3n-1)\frac{\pi}{3}$

These solutions are contained in the solution set of $\sin 3\theta = 0$.
Hence the required solution set is given by $\left\{\theta : \theta = \frac{n\pi}{3}, n \text{ an integer}\right\}$

Example 2.26
Solve

$$\sqrt{3}\cos\theta + \sin\theta = \sqrt{2}$$

Solution
Divide the equation by 2 to get

$$\frac{\sqrt{3}}{2}\cos\theta = \frac{1}{2}\sin\theta = \frac{1}{\sqrt{2}}$$

$$\text{or } \cos\frac{\pi}{6}\cos\theta + \sin\frac{\pi}{6}\sin\theta = \cos\frac{\pi}{4}$$

$$\text{or } \cos(\frac{\pi}{6} - \theta) = \cos\frac{\pi}{4} \text{ or } \cos(\theta - \frac{\pi}{6}) = \cos\frac{\pi}{4}$$

Hence, the solution is $\theta - \frac{\pi}{6} = 2m\pi \pm \frac{\pi}{4}$

or $\quad \theta = 2m\pi \pm \frac{\pi}{4} + \frac{\pi}{6}$

or $\quad \theta = 2m\pi - \frac{\pi}{4} + \frac{\pi}{6}$ or $2m\pi + \frac{\pi}{4} + \frac{\pi}{6}$

$\therefore \theta = 2m\pi - \frac{5\pi}{12}$

or $\quad \theta = 2m\pi - \frac{\pi}{12}$

Note: The above method can be used to solve equations of the form

$$a\cos\theta + b\sin\theta = c$$

Divide the equation by $\sqrt{a^2+b^2}$ to get

$$\frac{a}{\sqrt{a^2+b^2}}\cos\theta + \frac{b}{\sqrt{a^2+b^2}}\sin\theta = \frac{c}{\sqrt{a^2+b^2}}$$

Let $\tan\alpha = \frac{b}{a}$. Then

$$\sin\alpha \frac{b}{\sqrt{a^2+b^2}}, \cos\alpha = \frac{a}{\sqrt{a^2+b^2}}$$

The equation now becomes

$$\cos\alpha\cos\theta + \sin\alpha\sin\theta = \frac{c}{\sqrt{a^2+b^2}}$$

Hence, $\cos(\theta-\alpha) = \dfrac{c}{\sqrt{a^2+b^2}}$
This will have solutions if

$\left|\dfrac{c}{\sqrt{a^2+b^2}}\right| \le 1$, because, in this case, we can find $(\theta-\alpha)=\beta$ (say), so that

$$\cos\beta = \frac{c}{\sqrt{a^2+b^2}}.$$

'he equation can be easily solved now.

'xample 2.27
'olve

$$2\cos^2 t + 3\sin t = 0.$$

'olution
'he equation yields

$$
\begin{aligned}
& 2(1-\sin^2 t)+3\sin t &&= 0\\
\text{r}\quad & 2\sin^2 t - 3\sin t - 2 &&= 0\\
\text{r}\quad & (2\sin t+1)(\sin t-2) &&= 0
\end{aligned}
$$

'ence, either $2\sin t+1=0$ or $\sin t-2=0$. But this last situation cannot occur. Hence, $\text{n}\, t = -\frac{1}{2} = \sin\frac{7\pi}{6}$.
'herefore, the solution is $t = n\pi + (-1)^n\frac{7\pi}{6}$.

EXERCISE 2.7

'olve the following equations:

1. $\cos\theta+\cos 3\theta-2\cos 2\theta=0$
2. $\sin 2\theta+\cos\theta=0$
3. $\sec^2 2\theta = 1-\tan 2\theta$
4. $\sin\theta=\tan\theta$
5. $\sin 3\theta+\cos 2\theta=0$
6. $\sin x+\cos x=\sqrt{2}\cos A$
7. $4\cos\theta-3\sec\theta=\tan\theta$
8. $\sin m\theta+\sin n\theta=0$
9. $2\tan\theta-\cot\theta=-1$
10. $\cot^2 x+\dfrac{3}{\sin x}+3=0$

CHAPTER 3

Mathematical Induction

3.1 Introduction

The word 'induction' means the method of inferring a general statement from the vali of particular cases. We must be cautious here that in mathematics this kind of infere is not allowed, even when a huge list of particular cases have been verified. Mathemat induction is a principle by which one can conclude a statement for all positive integ after proving certain related propositions.

Let us see an example to explain the need for our caution.

We know that the numbers 13, 23, 43, 53, 73, etc. are prime numbers. And numbers 33, 63, 93, etc. are composite. From these particular cases we formulat general statement: A number of the form $10n + 3$ is prime if n is not divisible by 3 this a true statement?

Even if there are hundreds of particular cases where this is known to be true, cannot conclude that this general statement is true.

In fact this statement is not true in general when the number 143 is of the $10n + 3$ with $n = 14$, but it is not a prime.

We say that 143 is a counter example to the statement.

Even when we do not have a counter example, we cannot conclude that a gen statement is true simply because it has been found to be true in all its particular that have been verified. We can at the best say that it is a reasonable conjecture.

This raises the question: How shall we prove a general statement after that is kn to be true in some particular cases? We shall formulate in the next two sections, such method called the principle of mathematical induction.

3.2 Preparation for Induction

A notation: Consider the statements of the form:

1. n is divisible by 3.
2. The number $10n + 3$ is prime.
3. $2^n > n$.

All these are statements concerning the natural numbers $n = 1, 2, 3, \ldots$. We use the notations $P(n)$ or $P_1(n)$ or $P_2(n)$ etc. to denote such statements. When we give values for $n = 1, 2, \ldots$, we obtain particular statements. If in the statement $P(n)$, we substitute $n = 3$, the particular statement so obtained, is denoted by $P(3)$. Let us see some more examples.

Example 3.1
If $P(n)$ is the statement "$n(n+1)$ is even", then what is $P(4)$?

Solution
$P(4)$ is the statement "4(4+1) is even" i.e., "20 is even".

Example 3.2
If $P(n)$ is the statement "$n^3 + n$ is divisible by 3", is the statement $P(3)$ true? Is the statement $P(4)$ true?

Solution
$P(3)$ is "30 is divisible by 3". It is true.
$P(4)$ is "68 is divisible by 3". It is not true.

Example 3.3
Let $P(n)$ be the statement "$3^n > n$." Is $P(1)$ true?

Solution
$P(1)$ is the statement "$3^1 > 1$", i.e. "$3 > 1$", so $P(1)$ is true.

Example 3.4
Let $P(n)$ be the statement "$3^n > n$". What is $P(n+1)$?

Solution
$P(n+1)$ is the statement "$3^{n+1} > n+1$".

Example 3.5
Let $P(n)$ be the statement "$3^n > n$." If $P(n)$ is true, prove that $P(n+1)$ is true.

Solution
are given that $3^n > n$, and we wish to prove that $3^{n+1} > n+1$.
$3^{n+1} = 3.3^n > 3.n$.
$3n > n+1$ (because for every natural number n, $2n > 1$). So $3^{n+1} > n+1$.
proves that if $P(n)$ is true, then $P(n+1)$ is true.

EXERCISE 3.1

1. If $P(n)$ is the statement "$n(n+1)(n+2)$ is divisible by 12," prove that $P(3)$ and $P(4)$ are true, but that $P(5)$ is not true.
2. If $P(n)$ is the statement. "$n^2 > 100$," prove that whenever $P(r)$ is true, $P(r+1)$ is also true.
3. If $P(n)$ is the statement "$2^n \geq 3n$," and if $P(r)$ is true, prove that $P(r+1)$ is also true.
4. If $P(n)$ is the statement "$2^{3n} - 1$ is an integral multiple of 7," prove that $P(1), P(2)$ and $P(3)$ are true.
5. If $P(n)$ is the statement as in problem 4, and if $P(r)$ is true, prove that $P(r+1)$ is true.
6. Give an example of a statement $P(n)$ such that it is true for all n.
7. Give an example of a statement $P(n)$ such that $P(3)$ is true, but $P(4)$ is not true.

3.3 The Principle of Mathematical Induction

Now we are ready to state the principle of mathematical induction:

Let $P(n)$ be a statement such that

(a) $P(1)$ is true

(b) $P(r+1)$ is true whenever $P(r)$ is true.

Then $P(n)$ is true for all natural numbers n.

We shall illustrate this principle by numerous examples.

Example 3.6

Let $P(n)$ be the statement "$n^2 + n$ is even".

Then (a) $P(1)$ is the statement "2 is even". It is true.

(b) If $P(r)$ is true for some r, then to prove that $P(r+1)$ is true, consider

$$\begin{aligned}
(r+1)^2 + (r+1) &= r^2 + 2r + 1 + r + 1 \\
&= r^2 + r + 2(r+1) \\
&= \text{an even number } + 2(r+1) \qquad \text{(because } P(r) \text{ is true)} \\
&= \text{sum of two even numbers} \\
&= \text{an even number}
\end{aligned}$$

Thus $P(r+1)$ is proved to be true, assuming that $P(r)$ is true.

Therefore (since we have proved both the steps (a) and (b) required for the princip of induction), it follows by the principle of induction that $P(n)$ is true for all n. No

that the conclusion is so strong that it contains infinite number of statements one for each n.

Example 3.7

Let $P(n)$ be the statement "$n^3 + n$ is divisible by 3".
Here, $P(1)$ becomes "2 is divisible by 3". This is not true.
Therefore, the principle of induction does not apply.

Example 3.8

Let $P(n)$ be the statement "$n^2 > 100$."
Here, $P(1)$ is not true.

Therefore, the principle of induction does not apply. Note, however, that the second part namely 'if $P(r)$ is true, then $P(r+1)$ is true', can be proved here. Still, because $P(1)$ is not true, we conclude that $P(n)$ fails for some values of n.

Example 3.9

Prove that 7 divides $2^{3n} - 1$ for all positive integers.

Solution

Let $P(n)$ be the statement that 7 divides $2^{3n} - 1$.

Then (a) $P(1)$ is the statement "7 divides $2^3 - 1$". This is seen to be true.

(b) Suppose $P(r)$ is true. Then to prove that $P(r+1)$ is true.

$$\begin{aligned}
\text{Consider } 2^{3(r+1)} - 1 &= 2^{3r+3} - 1 \\
&= 2^{3r}.2^3 - 1 \\
&= 2^{3r}.8 - 1 \\
&= (2^{3r} - 1)8 + 8 - 1 \\
&= (\text{ a multiple of } 7) + 7 \qquad \text{(because } P(r) \text{ is true)} \\
&= \text{a multiple of } 7.
\end{aligned}$$

Therefore, by the principle of mathematical induction, $P(n)$ is true for all natural numbers n.

EXERCISE 3.2

Prove the following by the principle of induction:

1. The sum of the first n natural numbers is $\frac{n(n+1)}{2}$.

2. $n(n+1)(n+2)$ is divisible by 6, where n is a natural number.

3. $1 + 4 + 7 + \ldots + (3n-2) = \frac{n(3n-1)}{2}$.

4. If 3^{2n}, where n is a natural number, is divided by 8, the remainder is always 1.

5. $4 + 8 + 12 + \ldots + 4n = 2n(n+1)$.
6. If x and y are any two distinct integers, then $x^n - y^n$ is an integral multiple of $x - y$.
7. The sum $S_n = n^3 + 3n^2 + 5n + 3$ is divisible by 3 for any positive integer n.
8. $1^2 + 2^2 + \ldots\ldots + n^2 = \frac{1}{6}n(n+1)(2n+1)$ for every positive integer n.
9. $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots\ldots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$
10. If a set has n elements, prove that it has 2^n subsets.

MISCELLANEOUS EXERCISE ON CHAPTER 3

1. Prove by induction that the sum of first n odd natural numbers is n^2.
2. If we take any three consecutive natural numbers, prove that the sum of their cubes is always divisible by 9.
3. Prove by induction the inequality $(1+x)^n \geq 1 + nx$ whenever x is positive and n is a positive integer.
4. If $P(n)$ is the statement $n^2 - n + 41$ is prime, prove that $P(1)$, $P(2)$ and $P(3)$ are true. Prove also that $P(41)$ is not true. How does this not contradict the principle of induction?
5. Prove by induction that $2n + 7 < (n+3)^2$ for all natural numbers n. Using this, prove by induction that $(n+3)^2 < 2^{n+3}$ for all natural numbers n.
6. Prove that for $n \in N$
   $10^n + 3.4^{n+2} + 5$ is divisible by 9.
7. Prove that $10^{2n-1} + 1$ is divisible by 11 and for all $n \in N$.
8. Prove that
   $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}, n \in N$.
9. Prove that
   $1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$ for every positive integer n.

CHAPTER 4

Cartesian System of Rectangular Coordinates

4.1 Introduction

The geometry you have been studying thus far in earlier classes is called *Euclidean Geometry*. Its approach, as you would recall, was to start with certain *concepts*, like the concepts of points, lines and planes; attribute certain properties to them which we called *axioms* or *postulates* (suggested by physical experience); and then using the methods of deductive logic to derive a number of *theorems* which formed the main fruit of our mathematical activity and revealed to us the interesting and useful properties of the geometric figures under consideration. This approach was first presented by the Greek mathematician *Euclid*, around 300 BC, in his famous treatise "Elements" comprising thirteen books, and is being followed since then. As you would also recall, this made essentially no use of the process of algebra, and is called the *synthetic* approach to geometry.

This was the only approach to geometry for some two thousand years till the French philosopher and mathematician Rene Descartes (1596-1665) published *La Geometrie* in 1637 wherein he introduced the analytic approach (as against synthetic) by systematically using algebra in his study of geometry. This was achieved by representing points in the plane by ordered pairs of real numbers (called *Cartesian Coordinates* named after Rene Descartes), and representing lines and curves by algebraic equations. This wedding of algebra and geometry is known as *analytic* or *coordinate geometry*, and this is what we propose to study here.

4.2 Cartesian Coordinate System

In this section we shall establish a 1-1 correspondence between points on a straight line and the real numbers, and subsequently a 1-1 correspondence between points in the Euclidean plane and ordered pairs of real numbers. This would make it possible to apply the methods of algebra to study problems in geometry.

We are familiar with the representation of real numbers on a line, which we call the *real line*, or the number line, and denote by R^1 (or R). This was achieved through directed line segments and fixing a unit for length measurement. Fix a point O on the line,

which we shall call the *origin* from where all distances should be measured. This divides the line into two parts, the points on the left and right of the origin O. The distances measured (in terms of the fixed units) in the two parts are taken to be of opposite signs. This gives us the idea of directed line segments where not only length, but directions are also taken into account. If A is any other point on the line, then the line segment OA will be called directed line segment, directed from O to A. Obviously then, as directed line segments, $\overrightarrow{OA} = -\overrightarrow{AO}$. Distances measured to the right are conventionally taken as positive, and those measured to the left as negative. Thus every point P on this line corresponds to a real number x whose magnitude is the length OP measured in the prescribed units, and whose sign is +ve or −ve according as P is to the right or left of the origin O. Conversely, given a real number x we can always find a point P on the line on the right or left of O depending on the sign of x, such that the length OP equals $|x|$ units. This establishes a 1-1 correspondence between the points on the line and real numbers.

We now proceed to define a 1-1 correspondence between the points in the Euclidean plane and the set of all *ordered pairs of real numbers* (a, b). This can be done by defining what is called a *Cartesian Coordinate System* on the Euclidean plane, which we do as under:

In the Euclidean plane draw a horizontal line $X'OX$, a vertical line $Y'OY$ intersecting at O, the *origin*. We select a convenient unit of length and starting from the origin as zero, mark off a number scale on the horizontal line, positive to the right and negative to the left. We mark off the *same scale* on the vertical line, positive upwards and negative downwards of the origin O.

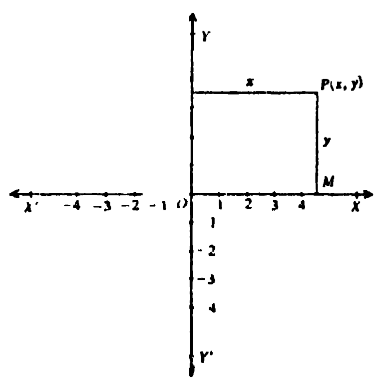


Fig 4.1

The horizontal line thus marked is called the *x-axis* and the vertical line the *y-axis*, and collectively they are called the *coordinate axes*.

Let P be any point in the plane. Draw perpendiculars from P to the coordinate axes, meeting the x-axis in M and the y-axis in N (Fig. 4.1). Let x be the length of the *directed line segment OM* in the units of the scale chosen. This is called the *x-coordinate* or *abscissa* of P. Similarly, the length of the directed line segment ON in the same scale is called the *y-coordinate* or *ordinate* of P. The position of the point P in the plane with respect to the coordinate axes is represented by the *ordered pair* (x, y) of real numbers, writing the abscissa first in the parenthesis. The pair (x, y) is called the *coordinates* of P, and this system of coordinating an ordered pair (x, y) with every point of the plane is called the (Rectangular) *Cartesian Coordinate System*.

We thus see that to every point P in the Euclidean plane there corresponds a unique

ordered pair (x, y) of real numbers called its Cartesian Coordinates. Conversely, given an ordered pair (x, y) and a cartesian coordinate system, we mark off a directed line segment $OM = x$ on the x-axis and another directed line segment $ON = y$ on y-axis, draw perpendiculars at M and N to x and y-axes respectively, and their point of intersection shall uniquely locate the corresponding point P in the Euclidean plane. This estabilishes a 1-1 correspondence between the set of all ordered pairs (x, y) of real numbers and the points in the Euclidean plane. The set of all ordered pairs (x, y) of real numbers is called Cartesian plane or simply plane and is denoted by R^2.

Finally we observe (Fig. 4.2) that the two axes divide the plane into four regions called the *quadrants*. The ray OX is taken as positive x-axis, OX' as negative x-axis, OY as positive y-axis and OY' as negative y-axis. The quadrants are thus characterised by the following signs of abscissa and ordinate.

I	quardant	$x > 0, y > 0$	or (+,+)
II	quardant	$x < 0, y > 0$	or (-,+)
III	quardant	$x < 0, y < 0$	or (-,-)
IV	quardant	$x > 0, y < 0$	or (+,-)

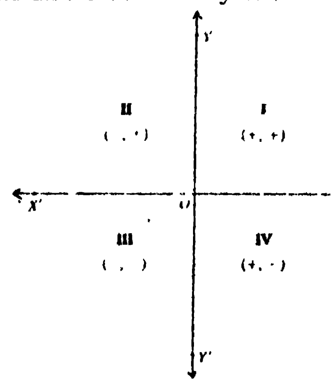


Fig.4.2

Further if the abscissa of a point is zero, it would lie somewhere on the y-axis and if its ordinate is zero it would lie on x-axis. Thus by simply looking at the coordinates of a point we can tell in which quadrant it would lie, e.g. the points (3, 4), (1, -2), (-2, -3) and (-4, 5) lie respectively in I, IV, III and II quadrants.

4.3 Distance Formula

The distance between any two points in the plane is the length of the line segment joining them. Let the coordinates of these two points P and Q be (x_1, y_1) and (x_2, y_2) respectively. We shall sometimes refer to them as points $P(x_1, y_1)$ and $Q(x_2, y_2)$ and obtain, as under, a formula for the distance between them.

Theorem 4.1

The distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Proof: Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the two points in the plane, and let d be the distance between them (Fig. 4.3). Draw lines parallel to y-axis from the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ which will meet the x-axis in points $A(x_1, 0)$ and $B(x_2, 0)$ respectively. Now draw a line through $P(x_1, y_1)$ parallel to x-axis which will meet the vertical through

Q in $R(x_2, y_1)$. The length of the segment between P and R, which we shall denote by $|PR|$, is equal to $|AB|$.

As in the figure, $|AB| = |OB| - |OA| = (x_2 - x_1)$. If, however, x_2 were to the left of x_1 (i.e. $x_2 < x_1$), this length were $(x_1 - x_2)$. In other case, since the length has got to be positive, we take the absolute value of $(x_2 - x_1)$, viz. $|x_2 - x_1|$ as the length $|AB|$. Hence, $|PR| = |AB| = |x_2 - x_1|$. In passing we observe that when the ordinates of two points (in this case P and R) are the same, the distance between them is the absolute value of the difference between their abscissa (in this case $|x_2 - x_1|$).

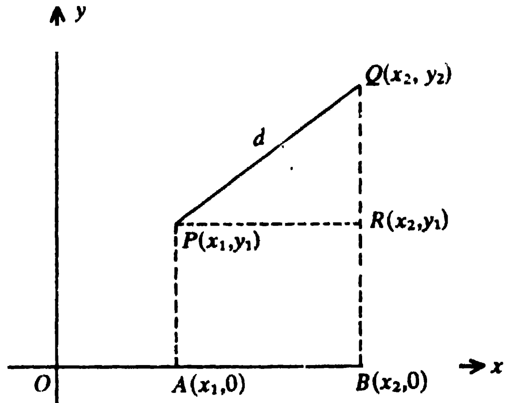


Fig.4.3

Repeating the same argument for the points Q and R, by drawing lines parallel to x-axis and meeting the y-axis, we shall find that the length $|RQ| = |y_2 - y_1|$. (What do you observe in this case? It would also be a good exercise to check the validity of these facts when P and Q lie in different quadrants).

Now applying Pythagoras theorem, we get

$$|PQ|^2 = |PR|^2 + |RQ|^2 \text{ or } d^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2$$

which can also be written as

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

Since the distance is always positive, taking the positive square root, we get the *distance formula*

$$d = |PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

This proves the theorem in case the line PQ is parallel to neither x-axis nor y-axis.

If PQ is parallel to x-axis, then obviously $y_1 = y_2$ and $PQ = |x_2 - x_1|$.

Also, $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(x_2 - x_1)^2} = |x_2 - x_1|$. Hence, the theorem is proved if PQ is parallel to x-axis. A similar proof can be given when PQ is parallel to y-axis.

Corollary: The distance of any point $P(x, y)$ from the origin is $\sqrt{x^2 + y^2}$.

Proof: In the above formula, take the point P as (x, y) and Q as $(0, 0)$, i.e. the origin, to get the result.

Example 4.1

What is the distance between the points (4, 5) and (–3, 2)?

Solution

It is immaterial whether we select (4, 5) or (-3, 2) as P, since the choice effects only the sign of $(x_2 - x_1)$ and $(y_2 - y_1)$. If we take (4, 5) as Q, then

$$\begin{aligned} PQ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{[4 - (-3)]^2 + (5 - 2)^2} \\ &= \sqrt{(4 + 3)^2 + (3)^2} \\ &= \sqrt{(7)^2 + (3)^2} \\ &= \sqrt{58} \end{aligned}$$

Example 4.2

Prove that the points (4, 4), (3, 5) and (-1,-1) represents the vertices of a right triangle.

Proof: Let the points (4, 4), (3, 5) and (-1, -1) represent the points P, Q and R respectively. Then,

$$\begin{aligned} PQ &= \sqrt{(3 - 4)^2 + (5 - 4)^2} = \sqrt{2} \\ QR &= \sqrt{[-1 - (3)]^2 + [-1 - (5)]^2} = \sqrt{52} \\ \text{and} \quad PR &= \sqrt{[-1 - (4)]^2 + [-1 - (4)]^2} = \sqrt{50} \end{aligned}$$

Since $QR^2 = RP^2 + PQ^2$, it follows from the converse of the Pythagoras Theorem that the triangle is a right triangle, with right angle at P.

EXERCISE 4.1

1. Find the distance between each of the following pairs of points:

   (i) $(-1, -4), (3, 5)$

   (ii) $(a\cos\alpha, a\sin\alpha)$, $(a\cos\beta, a\sin\beta)$

2. By using the distance formula, prove that each of the following sets of points are the vertices of a right triangle:

   (i) $(6, 2), (3, -1), (-2, 4)$

   (ii) $(-2, 2), (8, -2), (-4, -3)$

3. Show that each of the triangles whose vertices are given below are isosceles:

   (i) $(8, 2), (5, -3), (0, 0)$

   (ii) $(0, 6), (-5, 3), (3, 1)$

4. Find the value of x such that $PQ = QR$, where P, Q and R are (6, -1), (1, 3) and (x, 8), respectively.

5. What point on the x-axis is equidistant from (7, 6) and $(-3, 4)$?

6. Give the relation that must exist between x and y so that (x, y) is equidistant from $(6, -1)$ and (2, 3).

7. Show that the quadrilateral with vertices $(3, 2), (0, 5), (-3, 2), (0, -1)$ is a square.

4.4 Area of a Triangle

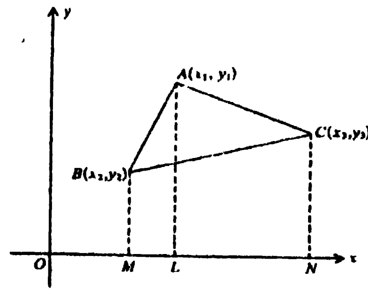


Fig.4.4

We now proceed to find the area of a triangle ABC, the coordinates of whose vertices are given to be $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$. Draw perpendiculars from A, B and C to x-axis meeting it in L, M and N respectively. As we have seen earlier, $|LM|$ or simply $ML = |x_1 - x_2| = x_1 - x_2$.

Similarly, $LN = (x_3 - x_1)$ and $MN = x_3 - x_2$. Now, the area of $\triangle ABC$ = area of trapezium $BMLA +$ area of trapezium $ALNC -$ area of trapezium $BMNC$ =

$$\begin{aligned}
&\frac{1}{2}(MB + AL).ML + \frac{1}{2}(AL + CN).LN - \frac{1}{2}(BM + CN).MN \\
&= \frac{1}{2}(y_2 + y_1)(x_1 - x_2) + \frac{1}{2}(y_1 + y_3)(x_3 - x_1) - \frac{1}{2}(y_2 + y_3)(x_3 - x_2) \\
&= \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)].
\end{aligned}$$

The expression

$$\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}$$

is sometimes written in short as

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Such an expression is called a *determinant*.
With this notation, the area of the triangle with vertices $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) is given by

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Note

1. Sometimes the above expression for area may turn out to be negative. But we take the absolute value of the expression as the area.
2. The above proof uses the fact that all vertices of the triangle are on one side of the y-axis; it can be shown that even if some vertex or vertices are on the other side of the y-axis the same expression will give us the area of the triangle.
3. We have given the above proof by drawing perpendiculars from vertices on the x-axis. A proof can also be given by drawing perpendiculars on the y-axis.

Condition of Collinearity of Three Points

Three points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are collinear, i.e. lie on the same straight line if and only if the area of the ΔABC is zero. Hence, three points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are collinear if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Example 4.3
Find the area of a triangle whose vertices are (4, 4), (3, –2), and (–3, 16).

Solution
Using the above formula, required area is

$$\begin{vmatrix} 4 & 4 & 1 \\ 3 & -2 & 1 \\ -3 & 16 & 1 \end{vmatrix} = \frac{1}{2}[4(-2-16) - 4(3-(-3)) + 1(48-6)] = -27$$

Rejecting the negative sign, area of triangle $= 27$ sq. units.
Note: If we actually plot the vertices and take them in anti-clockwise direction, the order would be (4, 4), (–3, 16) and (3, –2) and then the value of the determinant would be +27.

Example 4.4
Show that the three points (–1, –1), (2, 3) and (8, 11) lie on a line.
Proof: We have

$$1 \begin{vmatrix} -1 & -1 & 1 \\ 2 & 3 & 1 \\ 8 & 11 & 1 \end{vmatrix} = \frac{1}{2}[-1(3-11) + 1(2-8) + 1(22-24)] = 0$$

Hence the result.

EXERCISE 4.2

1. Find the area of the triangle with vertices at the points given in each of the problems (a) to (d).

   (a) (0, 0), (1, 0), (1, 1)
   (b) (-2, 1), (2, -3), (4, 4)
   (c) (3, 8), (-4, 2), (5, -1)
   (d) (2, 7), (3, -1), (-5, 6)

2. Show that each of the following triple of points are collinear:

   (a) (2, 4), (0, 1), (4, 7)
   (b) (-2, 5), (2, -3), (0, 1)
   (c) (-5, 7), (-4, 5), (1, -5)

3. For what value of x will the points $(x, -1)$, (2, 1) and (4, 5) lie on a line?
4. A and B are two points (3, 4) and (5, -2). Find the coordinates of a point P such that $PA = PB$ and the area of the triangle $PAB = 10$.
   (*Hint:* We can take the area as 10 or -10, hence two different points.)
5. Find the condition that the point (x, y) may lie on the line joining (3, 4) and (-5, -6).

4.5 Section Formula

Theorem 4.2
Given two points $A(x_1, y_1)$ and $B(x_2, y_2)$. The coordinates of the point P on AB which divides the line segment AB in the ratio $l : m$ (internally) are given by

$$x = \frac{mx_1 + lx_2}{l+m}, y = \frac{my_1 + ly_2}{l+m}$$

Proof: Draw lines parallel to y-axis from A, B and P meeting x-axis in C, D, and Q respectively. Draw lines parallel to x-axis from A and P meeting PQ and BD in E and R respectively. This being given that $\frac{AP}{PB} = \frac{l}{m}$. It is easily seen that the two right angled triangles APE and PBR are similar, and hence,

$$\frac{AP}{PB} = \frac{AE}{PR} = \frac{PE}{BR} = \frac{l}{m}$$

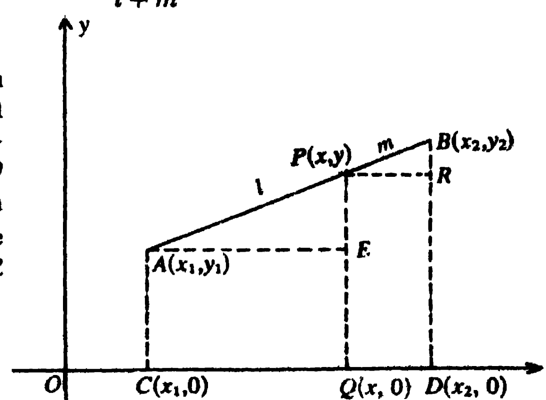


Fig.4.5

Now $AE = CQ = |OQ - OC| = |x - x_1| = x - x_1$
and $PR = QD = |OD - OQ| = |x_2 - x| = x_2 - x$

$$\text{Using } \frac{AP}{PB} = \frac{AE}{PR} = \frac{PE}{BR} = \frac{\ell}{m}$$

$$\frac{\ell}{m} = \frac{AE}{PR} = \frac{x - x_1}{x_2 - x} \text{ or } l(x_2 - x) = m(x - x_1)$$

$$\text{i.e. } x = \frac{mx_1 + lx_2}{l + m}$$

Again, $PE = |PQ - QE| = |PQ - AC| = |y - y_1| = y - y_1$
and $BR = |BD - RD| = |BD - PQ| = |y_2 - y| = y_2 - y$
Using $\frac{\ell}{m} = \frac{PE}{BR} = \frac{y - y_1}{y_2 - y}$, we have $l(y_2 - y) = m(y - y_1)$
i.e. $y = \dfrac{my_1 + ly_2}{l + m}$

Note: To remember the formula it is helpful to note that l is multiplied by the coordinate 'away from it', and similarly is m, and the sum then divided by $l + m$. This is diagrammatically shown in Fig. 4.6 as aid to memory.

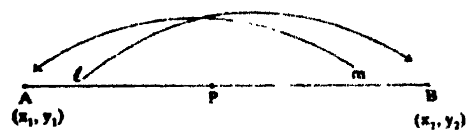


Fig.4.6

External Division

If the line AB is divided externally by a point P in the ration $l : m$ as shown in Fig. 4.7, then it is easy to see that $AP = l$ and $BP = m$, for a suitably chosen unit where P lies on AB produced.
Thus the point $B(x_2, y_2)$ divides the line AP *internally* in the ratio $(l - m) : m$. Our formula for internal division therefore implies that

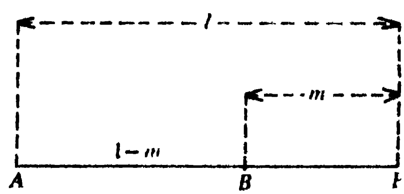


Fig.4.7

$$x_2 = \frac{(l - m)x + mx_1}{(l - m) + m}, y_2 = \frac{(l - m)y + my_1}{(l - m) + m},$$

so that

$$lx_2 = (l - m)x + mx_1 \text{ and } ly_2 = (l - m)y + my_1$$

giving

$$x = \frac{lx_2 - mx_1}{l - m}, y = \frac{ly_2 - my_1}{l - m}$$

Note that this is the same formula as for the internal division except that m is replaced by $-m$. If P divides AB externally in the ratio $l : m$ and $l < m$, then the coordinates of P will be given by

$$x = \frac{-lx_2 + mx_1}{-l + m}, y = \frac{-ly_2 + my_1}{-l + m}$$

Mid-Point Formula

To find the coordinates of the *mid-point* of a line segment with end points $A(x_1, y_1)$ and $B(x_2, y_2)$, we put $l = m$ in the formula of theorem 4.2 and obtain

$$x = \frac{x_1 + x_2}{2}, y = \frac{y_1 + y_2}{2}$$

Example 4.5

Find a point on the line through $A(5, -4)$ and $B(-3, 2)$, that is, twice as far from A as from B.

Solution

Obviously, there are two points A_1, A_2 that satisfy this requirement. A_1 divides AB internally in the ratio 2 : 1 and A_2 divides AB externally in the same ratio.

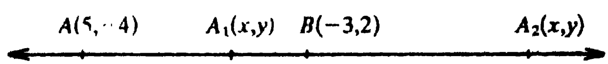


Fig.4.8

By theorem 4.2, coordinates of A_1 are

$$x = \frac{1(5) + 2(-3)}{1 + 2} = \frac{5 - 6}{3} = -\frac{1}{3}$$

$$y = \frac{1(-4) + 2(2)}{1 + 2} = \frac{0}{3} = 0$$

So coordinates of A_1 are $(-\frac{1}{3}, 0)$.
For A_2, we have

$$x = \frac{-1(5) + 2(-3)}{-1 + 2}, \qquad y = \frac{-1(-4) + 2(2)}{-1 + 2}$$

$$= \frac{-5 - 6}{1} = -11, \qquad \frac{4 + 4}{1} = 8$$

Therefore, A_2 has coordinates $(-11, 8)$.

Example 4.6

Find the centroid of the triangle whose vertices are $A(-1,0)$, $B(5,-2)$ and $C(8,2)$.

Solution

Centroid, the point where the medians of a triangle intersect, divides each median in the ratio 2:1. Coordinates of the mid point of BC are $\left(\frac{5+8}{2}, \frac{-2+2}{2}\right)$ i.e. $\left(\frac{13}{2}, 0\right)$

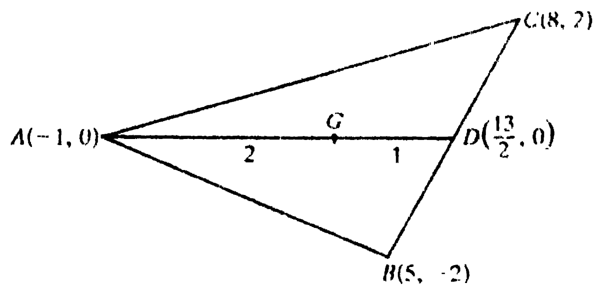


Fig 4.9

If G is the centroid, it must divide the median AD in the ratio $2:1$. Hence, by section formula, its coordinates are

$$x = \frac{2 \times \left(\frac{13}{2}\right) + 1(-1)}{2+1} = 4, \quad y = \frac{2 \times 0 + 1 \times 0}{2+1} = 0$$

Hence, the coordinates of the centroid G are $(4, 0)$. The reader is advised to check the result corresponding to other two medians.

EXERCISE 4.3

1. Find the coordinates of the points which divide internally and externally the line joining $(1,-3)$ and $(-3,9)$ in the ratio 1:3.
2. Prove that the points $(-2,-1), (1,0), (4,3)$ and $(1,2)$ are the vertices of a parallelogram.
   (*Hint:* Diagonals of a parallelogram bisect each other)
3. Find the centroid of a triangle whose vertices are $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3).

4. Find the third vertex of a triangle if two of its vertices are at $(-1,4)$ and $(5,2)$ and the centroid at (0, -3).

5. In what ratio does the point $(\frac{1}{2}, 6)$ divide the line segment joining the points (3, 5) and $(-7, 9)$?

6. Find the ratio in which the line segment joining (2, -3) and (5, 6) is divided by x-axis.

4.6 Slope of a Line

A line l not parallel to x-axis in a coordinate plane intersects it. Such a line forms two angles with x-axis which are supplementary. To be definite, we choose that angle x which is made going anticlockwise direction from x-axis. This angle x will have values between 0° and 180°, and is called the *angle of inclination or inclination* of the line l. All lines parallel to x-axis, or coinciding with x-axis, are said to have inclination 0°. Note that when two lines are parallel, they have the same inclination.

For the purpose of analytic geometry we associate a number with the inclination of the line in the following manner:

Definition 4.1
The slope m of a line having inclination α, and not perpendicular to x-axis, is defined to be $\tan\alpha$.

The slope of a line perpendicular to x-axis is not defined as the value of $\tan\alpha$ at $\alpha = 90°$ is undefined.

Note that the slope m is independent of the sense of line segment. As shown in Fig. 4.10 slope of $AB = \tan\alpha = \tan(\pi + \alpha)$ = slope of BA and hence we do not take into consideration the direction of a line segment, while talking of its slope.

We know that a line is fully determined when we are given two points on it. Hence, we proceed to find a formula for the slope of a line in terms of the coordinates of two points given on it.

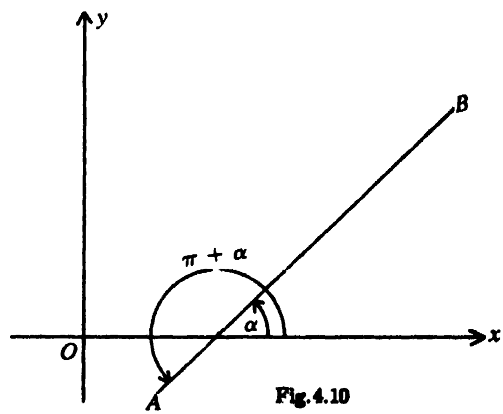


Fig.4.10

Theorem 4.3
If $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two points on a line l, then the slope m of the line l is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}, \; x_1 \neq x_2$$

If $x_1 = x_2$, then m is not defined. In that case the line is perpendicular to x-axis.

Proof: Let α be the inclination of the line l. We shall consider two different cases when α is acute or obtuse, as shown in Fig. 4.11(i) and (ii). Draw horizontal and verticle lines

(i.e. parallel to x-axis and y-axis respectively) through P and Q respectively intersecting at M, whose coordinates are shown in the figure.

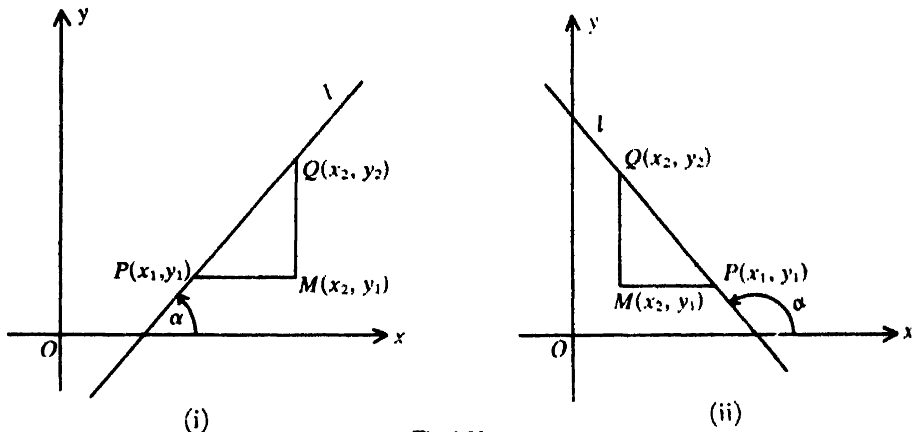


Fig.4.11

In Fig. 4.11(i) the inclination α is equal to $\angle MPQ$, hence $m = \tan\alpha = \tan\angle MPQ = \dfrac{MQ}{PM} = \dfrac{y_2 - y_1}{x_2 - x_1}$

In Fig. 4.11(ii) the inclination α and $\angle MPQ$ are supplementary, hence

$$m = \tan\alpha = \tan(180° - \angle MPQ) = -\tan\angle MPQ = -\frac{MQ}{MP} = -\frac{y_2 - y_1}{x_1 - x_2} = \frac{y_2 - y_1}{x_2 - x_1}$$

Consequently, we see that in all cases the slope m of the line through points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

The difficulty arises when $x_1 = x_2$, i.e. when m is not defined. This is the case when the line l is parallel to y-axis or perpendicular to x-axis.

Thus given any line not perpendicular to x-axis, we can always find its slope by locating two points on it, and also given any point we can draw a line of the given slope through it.

Example 4.7

Find the slope of a line l determined by the points $P(4,6)$ and $Q(2,12)$.

Solution

Here $\quad m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{12 - 6}{2 - 4} = -3$

Obviously l makes an obtuse angle with x-axis.

Example 4.8

Through $P(4,1)$ construct a line which has slope equal to $\frac{3}{2}$.

Solution

Starting at P, draw a line parallel to x-axis extending to a point R one unit to the right (Fig. 4.12). Now draw RQ parallel to y-axis such that $RQ = \frac{3}{2}(= m)$ units above R. The coordinates of this point Q are $(5, \frac{5}{2})$

Hence, the slope $= \dfrac{\frac{5}{2} - 1}{5 - 4} = \dfrac{3}{2}$.

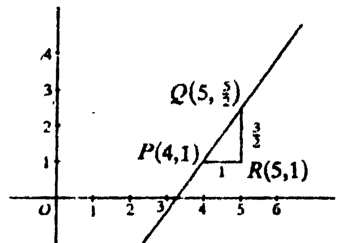


Fig.4.12

Parallel and Perpendicular Lines

If two lines are parallel (neither of which is parallel to y-axis), then their inclinations are same and hence their slopes are also same. Conversely, if the slope m of two lines is the same, then by the property of tangent function there exists a unique angle x between 0° and 180° such that $\tan x = m$, and hence their inclinations are same which means they are parallel. Thus we arrive at the following result:

Theorem 4.4

Two lines (not parallel to y-axis) are parallel if and only if their slopes are equal.

The question arises: Can we characterise perpendicularity also in terms of slope? The answer is given by the following theorem:

Theorem 4.5

Two lines (non parallel to y-axis) are perpendicular, if and only if their slopes m_1, m_2 satisfy the condition that $m_1 m_2 = -1$.

Proof:

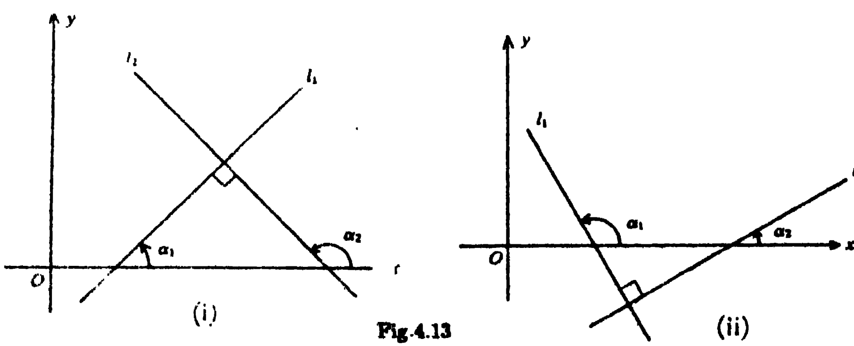


Fig.4.13

As in Fig. 4.13 (i) and (ii), if the lines ℓ_1 and ℓ_2 are perpendicular, then α_1 and α_2 differ by 90°, i.e. $\alpha_2 = \alpha_1 \pm 90°$, hence $\tan\alpha_2 = \tan(\alpha_1 \pm 90°)$. In either case we get

$$m_2 = \tan\alpha_2 = -\cot\alpha_1 = -\frac{1}{\tan\alpha_1} = -\frac{1}{m_1}$$

$$\text{or} \qquad m_1 m_2 = -1$$

Conversely, if $m_1 m_2 = -1$ i.e. $\tan\alpha_1 \tan\alpha_2 = -1$, then $\tan\alpha_2 = -\cot\alpha_1 = \tan(90° + \alpha_1)$ or $\tan(\alpha_1 - 90°)$ which is satisfied when α_1 and α_2 differ by 90° i.e. ℓ_1 and ℓ_2 are perpendicular.

Remark: If the slope of one of them is undefined, it is parallel to y-axis and so the perpendicular line has slope zero and is parallel to x-axis.

Example 4.9
Show that the line through (0, 0) and (2, 3) is parallel to the line through (2, -2) and (6, 4).
Proof:

$$m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-0}{2-0} = \frac{3}{2}$$
$$m_2 = \frac{4-(-2)}{6-2} = \frac{6}{4} = \frac{3}{2}$$

Since the slopes are equal, the lines are parallel.

Example 4.10
Prove that the line through (-2,6) and (4,8) is perpendicular to the line through (8,12) and (4,24).
Proof:

$$m_1 = \frac{8-6}{4-(-2)} = \frac{1}{3} \quad \text{and} \quad m_2 = \frac{24-12}{4-8} = -3.$$

Hence, $m_1 m_2 = -1$ and the lines are perpendicular.

EXERCISE 4.4

1. What can be said regarding a line if its slope is
   (a) positive, (b) zero, (c) negative?
2. What is the slope of a line whose inclination is
   (a) 0°, (b) 45° (c) 90° (d) 150°

3. Find the slope of the line through the points
   (a) (1, 2), (4, 2) (b) (0, -4), (6, 2) (c) (4, -6), (-2, -5)
4. Show that the line joining (2, 3) and (-5, 1) is
   (a) parallel to the line joining (7, -1) and (0, 3)
   (b) perpendicular to the line joining (4, 5) and (0, 2)
5. State whether the two lines in each of the following are parallel, perpendicular or neither.
   (a) Through (5, 6) and (2, 3); through (9, -2) and (6, 5)
   (b) Through (8, 2) and (5, 3); through (16, 6) and (3, 15)
   (c) Through (2, 5) and (-2, 5); through (6, 3) and (1, 1)
   (d) Through (9, 5) and (1, 1); through (8, -3) and (3, -5)
6. Determine x so that 2 is the slope of the line through (2, 5) and (x, 3).
7. What is the value of y so that the line through (3, y) and (2, 7) is parallel to the line through (1, 1) and (0, 6)?
8. Without using Pythagoras Theorem, show that (4, -4), (3, 5) and (1, 1) are the vertices of a right triangle.
9. A quadrilateral has the vertices at the points (-4, 2), (2, 6), (8, 5) and (9, -7). Show that the mid-points of the sides of this quadrilateral are the vertices of a parallelogram.

4.7 Sets of Points and Equations

Let us consider the following:

Consider a circle of radius a whose centre is at the origin. This circle is the set of all points in the plane whose distance from the origin is a

Let $P(x, y)$ be any point on this circle. Then as its distance from origin (0, 0) is a, we have

$$\sqrt{x^2 + y^2} = a$$

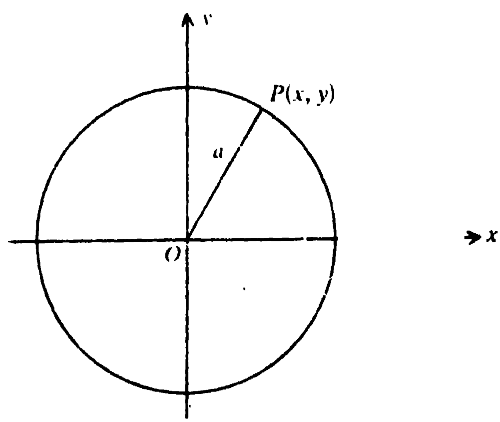


Fig.4.14

and so, for every point $P(x, y)$ on the circle, we get $x^2 + y^2 = a^2$.

Conversely, if (x, y) is any point in the plane satisfying the equation $x^2+y^2 = a^2$, then this shows that the distance of the point (x, y) from the origin is a, so that the point is on the circle. Thus the circle with centre as origin and radius a is the set of all points (x, y) such that $x^2 + y^2 = a^2$.

We say that $x^2 + y^2 = a^2$ is the equation of this circle.

Again, consider the line l parallel to x-axis and lying 4 units above the x-axis.
It is obvious that if $P(x, y)$ is any point on this line, then $y = 4$. Conversely any point (x, y) for which $y = 4$ will lie on this line l. Hence

$$l = \{(x, y) | y = 4\}.$$

So the equation of the line l is $y = 4$.
These examples indicate that if we take a set of points in the plane such as a line or circle, etc. we can find an equation in x and y (or only x or only y) such that a point (x, y) belongs to this set if and only if the coordinates (x, y) satisfy the equation.
In general, different sets of points will have different equations.

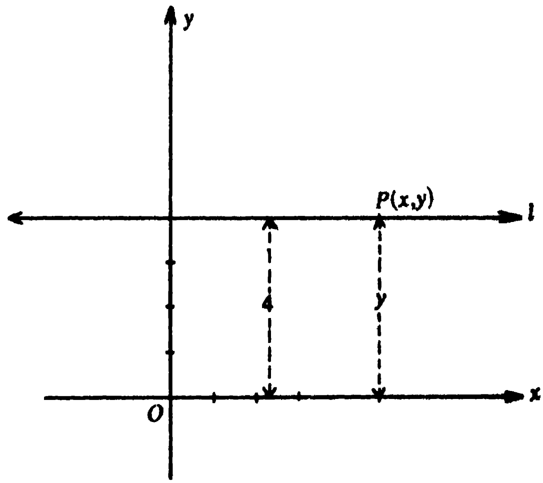


Fig.4.15

Example 4.11

Find the equation of the set of all points which are twice as far from (3, 2) as from (1,1).

Solution

Let A be the point (3, 2) and B the point (1, 1).
Suppose

$$S = \{P | PA = 2PB\}$$

Let $P \in S$ and P have coordinates (x, y).
As then

$$PA^2 = (x-3)^2 + (y-2)^2$$
$$PB^2 = (x-1)^2 + (y-1)^2$$

Now $\qquad PA = 2PB$

So $\qquad PA^2 = 4PB^2$

$$(x-3)^2+(y-2)^2 = 4\left\{(x-1)^2 + (y-1)^2\right\}$$

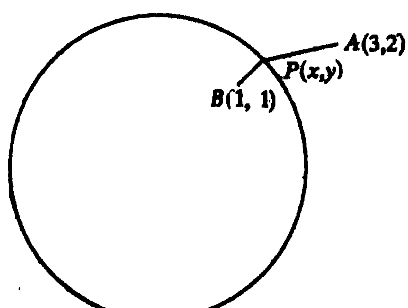


Fig.4.16

which, on simplification becomes

$$3x^2 + 3y^2 - 2x - 4y - 5 = 0$$

Example 4.12

Find the equation of the set of points such that the sum of its distances from (0,3) and (0, −3) is 8.

Solution

Let $P(x, y)$ be any point of the set and A and A' be the points (0,3) and (0,−3) respectively (See Fig. 4.17). We are given that
$|PA| + |PA'| = 8$. Using this, we have

$$\sqrt{(x-0)^2 + (y-3)^2} + \sqrt{(x-0)^2 + (y+3)^2} = 8$$

or $\quad x^2 + (y^2 - 6y + 9) = 64 - 16\sqrt{x^2 + (y+3)^2} + x^2 + y^2 + 6y + 9$

or $\quad -12y = 64 - 16\sqrt{x^2 + (y+3)^2}$

or $\quad 12y + 64 = 16\sqrt{x^2 + (y+3)^2}$

or $\quad 3y + 16 = 4\sqrt{x^2 + (y+3)^2}$

or $\quad 9y^2 + 96y + 256 = 16(x^2 + y^2 + 6y + 9)$

or $\quad 112 = 16x^2 + 7y^2.$

Hence, $\quad \frac{x^2}{7} + \frac{y^2}{16} = 1$

Therefore, the required equation is $\frac{x^2}{7} + \frac{y^2}{16} = 1$.

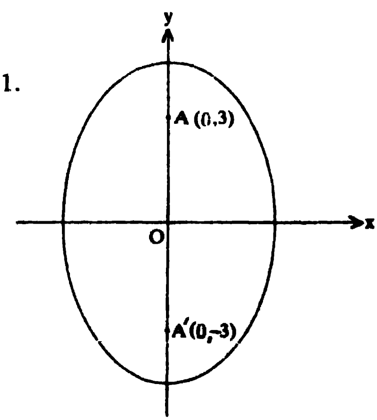


Fig.4.17

From the above discussion it is clear that if a coordinate system is defined, the condition determining a particular set may be expressible as an equation or a set of equations involving the coordinates x and y of a point. When it is possible, as is often the case, a relationship is established between sets and equations. This relationship is the basis of two fundamental problems in coordinate geometry.

1. Given a set (geometric condition), to find the corresponding algebraic relation (equation).

2. Given an algebraic relation (equation), to find the corresponding set.

In the articles that follow, we shall consider these two problems with respect to the sets — the straight lines and circles.

EXERCISE 4.5

1. Find the equation of the set of points equidistant from $(-1, -1)$ and $(4, 2)$.
2. Find the equation of the set of all points equidistant from the point $(4,2)$ and the x-axis.
3. Find the equation of the set of all points $P(x, y)$ such that the segment OP has slope 3, where O is the origin.
4. Find the equation of the set of points for which every ordinate is greater than the corresponding abscissa by a given distance.
5. Find the equation of the set of points such that the sum of their distances from $(0,2)$ and $(0,-2)$ is 6.
6. Find the equation of the set of all points $P(x, y)$ such that the line OP is coincident with the line joining P and the point $(3, 2)$.

CHAPTER 5

Straight Line

5.1 To Find the Equation of a Straight Line Parallel to an Axis

We know that on x-axis, the y-coordinates of all points are zero. We say that the equation of x-axis is $y = 0$. Similarly, the equation of y-axis is $x = 0$. Now the equation of a straight line parallel to x-axis is $y = b$, where b is the ordinate of any point on the line. Similarly, $x = a$ is the equation of a straight line parallel to y-axis, where a is the abscissa of any point on the line.

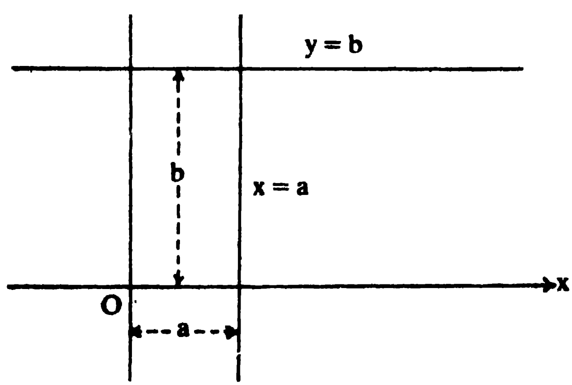


Fig.5.1

5.2 The Point-Slope Form

Now we are in a position to obtain the equation of a line determined by a given set of conditions.

Let $P_1(x_1, y_1)$ be a fixed point and m be a given slope. If any other point on the line is $P(x, y)$, then $\frac{y-y_1}{x-x_1}$ is the slope of the line through P_1 and P. But this is given as m. Hence,

$$\frac{y-y_1}{x-x_1} = m$$

$$\text{or} \quad y - y_1 = m(x - x_1) \tag{5.1}$$

Conversely, if a point $Q(x, y)$ in the plane satisfies (5.1), then as $\frac{y-y_1}{x-x_1}$ is the slope of QP_1, (5.1) expresses the fact that the slope of QP_1 is m. Thus Q is on a line through P_1 with slope m.

Thus, if l is the line through P_1 with slope m, then we have shown that

$$l = \{(x, y) | y - y_1 = m(x - x_1)\}$$

Hence the equation (5.1) is the equation of the line through the point (x_1, y_1) and slope m. This form of the equation of a line is called the point-slope form.

Note that the slope m is undefined for the lines parallel to y-axis. Hence, the point-slope form of the equation will not be applicable to the equation of a line through $P_1(x_1, y_1)$ parallel to the y-axis. However, this presents no difficulty, since for any such line the x-coordinate of any point on it is x_1, the equation of such a line is $x = x_1$.

Example 5.1

Determine the equation of the line passing through the point (-1, -2) and with slope $\frac{4}{7}$.

Solution

Putting the values $x_1 = -1, y_1 = -2$ and $m = \frac{4}{7}$ in the point-slope form of the equation, we get

$$
\begin{aligned}
& y - (-2) &= \frac{4}{7}[x - (-1)] \\
\text{or} \quad & y + 2 &= \frac{4}{7}(x + 1) \\
\text{or} \quad & 7y + 14 &= 4x + 4
\end{aligned}
$$

Therefore, equation of the line is $y = \frac{4}{7}x - \frac{10}{7}$

Example 5.2

Determine the equation of the line through the point (3, -4) and parallel to the x-axis.

Solution

A line parallel to the x-axis has the slope zero. Therefore, point-slope form of the equation gives

$$
\begin{aligned}
y - (-4) &= 0(x - 3) \\
\text{or} \quad y + 4 &= 0
\end{aligned}
$$

Another way to approach this problem is to note that every point on the line must have the same ordinate. Since one point has ordinate −4, we must have $y = -4$ for all points.

5.3 Two-Point Form

Let $Q_1(x_1, y_1)$ and $Q_2(x_2, y_2)$ be two points and let l be the line through these two points. If $x_1 = x_2$ then Q_1Q_2 is parallel to x-axis and the equation of l is $x = x_1$.

If $x_1 \neq x_2$, let $P(x, y) \in l$. Then as PQ_1 and Q_2Q_1 are the same lines they have the same slope. So

$$\frac{y_2 - y_1}{x_2 - x_1}$$

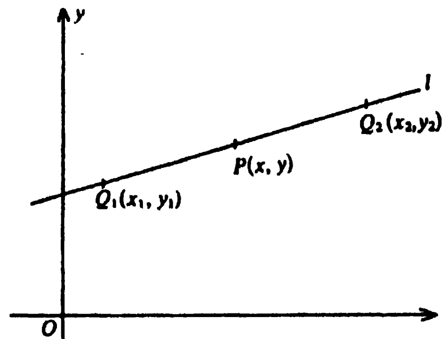


Fig.5.2

Conversely, if a point $Q(x, y)$ satisfies $\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$, then this last equation indicates that the slope of QQ_1 = slope of Q_2Q_1. So the lines QQ_1 and Q_1Q_2 are either the same or they are parallel. But these lines already have a common point Q_1, so they are the same lines. Thus Q is on Q_1Q_2, i.e. $Q \in l$. Hence

$$= \left\{ (x, y) \,\middle|\, \frac{y - y_1}{x - x_1} = \frac{y_1 - y_2}{x_1 - x_2} \right\}$$

Therefore equation of the line through (x_1, y_1) and (x_2, y_2) is

$$\frac{y - y_1}{x - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$$

or (5.2)

5.4 Intercept Form

Suppose a line l is parallel to neither x-axis nor y-axis. Then l intersects x-axis at some point $A(a, 0)$ and it intersects the y-axis at some point $B(0, b)$. We say that a and b are respectively x-intercept and y-intercept of l. Since l passes through the points $(a, 0)$ and $(0, b)$, we see that the two point form tells us that the equation of l is

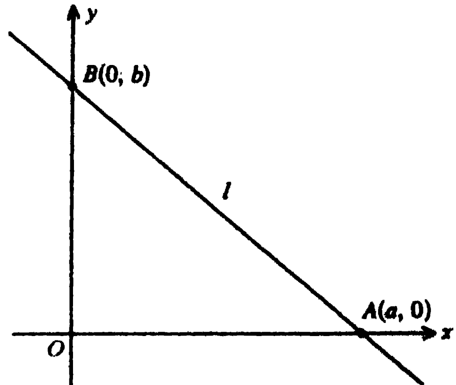


Fig.5.3

$$\frac{y-b}{b-0} = \frac{x-0}{0-a}, \text{ i.e. } \frac{y}{b} - 1 = -\frac{x}{a}$$

or
$$\frac{x}{a} + \frac{y}{b} = 1.$$

Thus the equation of a line whose x and y-intercepts are a and b is

$$\frac{x}{a} + \frac{y}{b} = 1 \tag{5.3}$$

5.5 Slope-Intercept Form

If a line is not parallel to y-axis, it may be determined by its y-intercept and slope m. If a line has the y-intercept b, it passes through the point $(0, b)$. Hence, we may use the point-slope form to obtain the equation of the line in terms of these quantities. Substituting $(0, b)$ for (x_1, y_1) in the point-slope form, we get

$$y - b = m(x - 0)$$

or
$$y = mx + b$$

This is the equation of the line with slope m and y-intercept b. This form of the equation of the straight line is very useful.

Alternative Method

Let the given intercept of a line l with y-axis be c and inclined at an angle α with x-axis.

Let C be the point on y-axis such that $OC = b$.
Through C draw a straight line inclined at an angle $\alpha[= \tan^{-1}(m)]$ to x-axis so that $\tan\alpha = m$. Let P be any point on the line. Draw PM perpendicular to x-axis to meet a line through C parallel to x-axis in N.

Let (x, y) be the coordinates of P. Therefore $OM = x$ and $MP = y$

$$\angle PCN = \angle PL'M = \alpha$$

$$MP = NP + MN$$

$$\text{or } NM = MP - NM = MP - OC$$

$$\text{or } NP = y - b \text{ and } CN = OM = x$$

In ΔPCN,

$$\tan\alpha = \frac{PN}{CN}$$

$$\text{or } m = \frac{y-b}{x}$$

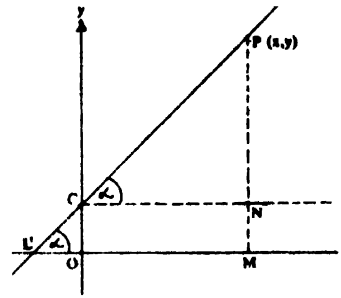


Fig.5.4

$$\text{or} \qquad y = mx + b \tag{5.4}$$

This is the equation of the line with slope m and y-intercept b.

Example 5.3

What is the equation of a line with slope 3 and y-intercept 2?

Solution

On substituting $m = 3$ and $b = 2$, in the slope-intercept form of the equation, we get $y = 3x + 2$.
This is the desired equation.

Example 5.4

Determine the slope and the y-intercept of the line whose equation is $5x + 6y = 7$.

Solution

Expressing y in terms of x we have

$$y = -\frac{5}{6}x + \frac{7}{6}$$

Comparing this equation with the slope-intercept form, we see that $m = -\frac{5}{6}$ and $b = \frac{7}{6}$.
Therefore, slope of the line is $-\frac{5}{6}$ and its y-intercept is $\frac{7}{6}$.

5.6 Normal Form

A straight line is determined if the length of the perpendicular from the origin (0,0) to the line, and the angle which this perpendicular makes with the x-axis are known.

Let AB be the line. Draw OP perpendicular to AB as shown in Fig. 5.5. Four different figures [See Fig. 5.5 (i), (ii), (iii), (iv)] are given for various positions of the line AB. Consider the first Fig. 5.5 (i).

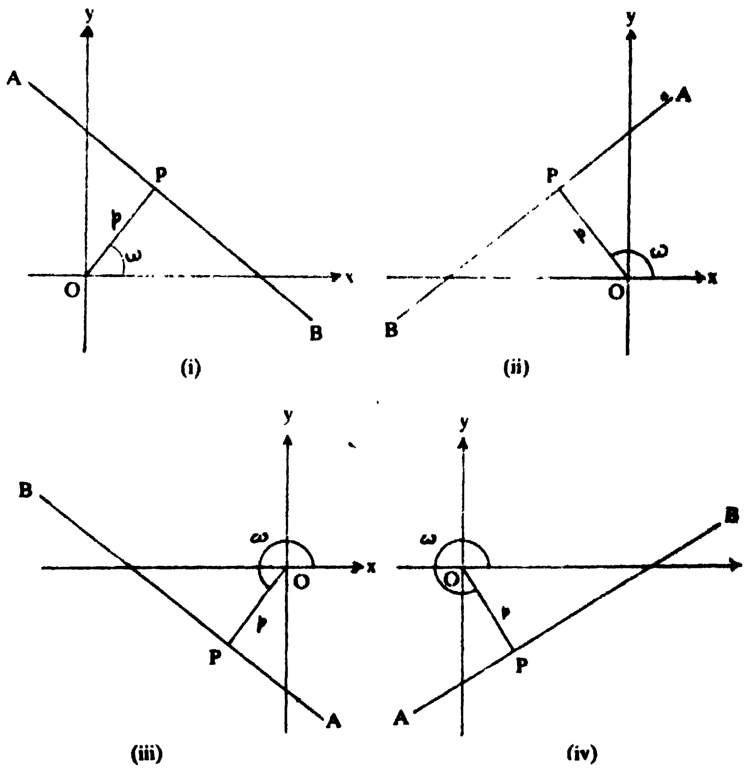


Fig.5.5

Let ω be the angle between OP and the positive x-axis, and p the length of the perpendicular OP. Then the coordinates of the point p will be $(p\cos\omega, p\sin\omega)$, and the slope of AB will be $-\frac{1}{\tan\omega} = -\cot\omega$. If (x, y) is any other point on the line AB, then by the point-slope formula, $y - p\sin\omega = -\cot\omega(x - p\cos\omega)$. On simplifying, we get

$$x\cos\omega + y\sin\omega - p = 0$$

or $$x\cos\omega + y\sin\omega = p \tag{5.5}$$

which is the perpendicular form of the straight line. It can be verified that we get the same form, if we consider Fig. 5.5 (ii) (iii) and (iv).

Example 5.5

Find the equation of the line with $\omega = 135^\circ$ and perpendicular distance 4.

Solution

From (5.5), we have

$$
\begin{aligned}
& & x\cos 135^\circ + y\sin 135^\circ - 4 &= 0 \\
\text{or} & & -\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} - 4 &= 0 \\
\text{or} & & -x + y - 4\sqrt{2} &= 0
\end{aligned}
$$

Example 5.6

Find the equation of the line which has length of perpendicular segment from the origin to the line as 4 and the inclination of the perpendicular segment with the positive direction of x-axis is 30°.

Solution

The normal form of the equation of a line is $x\cos\omega + y\sin\omega = p$
Here $p = 4$ and $\omega = 30^\circ$
$\therefore$ Equation of the line is

$$
\begin{aligned}
& & x\cos 30^\circ + y\sin 30^\circ &= 4 \\
\text{or} & & x\frac{\sqrt{3}}{2} + y\frac{1}{2} &= 4 \\
\text{or} & & \sqrt{3}x + y &= 8
\end{aligned}
$$

5.7 Symmetric Form

Let a line pass through $A(x_1, y_1)$ and be inclined at an angle θ with the positive direction of x-axis. Then the equation of the line involving x_1, y_1 and θ is called the symmetric form of the line.

Let $P(x,y)$ be any point and $AP = r$. Draw AB and PC perpendiculars to x-axis from A and P respectively and $AN \perp PC$.

Now, $AN = BC = OC - OB = x - x_1$
$PN = PC - CN = PC - AB = y - y_1$
Also $AP = r$

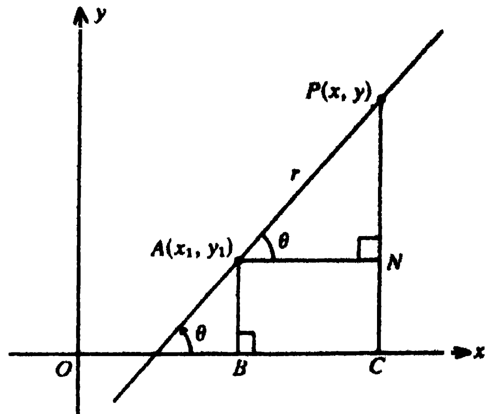


Fig.5.6

In ΔANP, $\cos\theta = \frac{AN}{AP} = \frac{(x - x_1)}{r}$

i.e. $\frac{x - x_1}{\cos\theta} = r$ (i)

and $\sin\theta = \frac{PN}{AP} = \frac{y - y_1}{r}$

i.e. $\frac{y - y_1}{\sin\theta} = r$ (ii)

From (i) and (ii)

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} \quad (5.6)$$

which is the equation of the line in the symmetric form.

Note: From the equation it follows that

$$x = x_1 + r\cos\theta, \quad y = y_1 + r\sin\theta, \quad (\theta = \text{constant})$$

which is called the parametric equation of the line, r being the parameter.

Example 5.7

Find the equation of a line which passes through the point (-2, 3) and makes an angle of 30° with the positive direction of x-axis.

Solution

Here $\theta = 30°$.

Given point on the line is (-2, 3). Using the symmetric form, the equation of the line is

$$\frac{x+2}{\cos 30°} = \frac{y-3}{\sin 30°} \quad (5.7)$$

or $$\frac{x+2}{\frac{\sqrt{3}}{2}} = \frac{y-3}{\frac{1}{2}}$$

or $$\sqrt{3}y - 3\sqrt{3} = x + 2$$

or $$x - \sqrt{3}y + (3\sqrt{3} + 2) = 0$$

is the equation of the required line.

5.8 General Form

All the forms in which we have found the equation of the straight line are of the first degree in x and y. The converse of this is also true. The most general form of any equation of the first degree in x and y is $Ax + By + C = 0$.

This equation, though apparently involving three constants A, B and C, in reality involves only two — namely, the ratios $\frac{A}{C}$ and $\frac{B}{C}$.

We shall now show that every straight line is represented by an equation of the first degree and conversely.

Theorem 5.1

The equation $Ax + By + C = 0$ always represents a straight line provided A and B are not zero simultaneously.

Proof: We shall now consider the following two cases:

Case I: If $B = 0$ and $A \neq 0$, then the equation $Ax + By + C = 0$ becomes $Ax + C = 0$ or $x = -\frac{C}{A}$ and is satisfied by all points lying on a line parallel to y-axis and at a distance $-\frac{C}{A}$ units from it. Hence, this is the equation of a straight line. The case where $B \neq 0$, and $A = 0$, can be treated similarly.

Case II: If $B \neq 0$ and $A \neq 0$, we can solve the equation for y and obtain

$$y = -\frac{A}{B}x - \frac{C}{B}$$

This represents the straight line with slope $-\frac{A}{B}$ and y-intercept $-\frac{C}{B}$

The converse is given in the following theorem.

Theorem 5.2

Every straight line has an equation of the form $Ax + By + C = 0$, where A, B and C are constants.

Proof: Given a straight line, either it cuts the y-axis, or is parallel to or coincident with it. We know that the equation of a line which cuts the y-axis (that is it has a y-intercept) can be put in the form $y = mx + b$; further, if the line is parallel or coincident with the y-axis, its equation is of the form $x = x_1$, where $x_1 = 0$ in the case of coincidence. Both of these equations are of the form given in the theorem; hence the proof.

We can use the form $Ax + By + C = 0$ to determine the equation of a straight line in the following way:

Example 5.8

Find the equation of the line through (3,4) and (2,–1).

Solution

We seek the numbers A, B and C such that the line $Ax + By + C = 0$ passes through the two given points. If $Ax + By + C = 0$ passes through (3,4), then $3A + 4B + C = 0$; if it passes through $(2, -1)$, then $2A - B + C = 0$.

Subtracting $2A - B + C = 0$ from $3A + 4B + C = 0$, we get $A + 5B = 0$.

So $A = -5B$; also $2A - B + C = 0$ now yields $-10B - B + C = 0$, i.e. $C = 11B$.

Thus the equation $Ax + By + C = 0$ becomes

$$-5Bx + By + 11B = 0$$

or $\quad -5x + y + 11 = 0$

or $\quad y = 5x - 11$

Therefore, the equation of the line through (3, 4) and (2, −1) is $y = 5x - 11$.
The general equation of a straight line is reducible to the normal form in the following way
The general equation of the straight line is

$$Ax + By + C = 0 \tag{5.8}$$

The equation of a line in the normal form is

$$x\cos\alpha + y\sin\alpha - p = 0 \tag{5.9}$$

If we suppose that (5.8) and (5.9) represent the same straight line, then we can compare the corresponding coefficients

$$\frac{A}{\cos\alpha} = \frac{B}{\sin\alpha} = \frac{C}{-p}$$

or $\quad \cos\alpha = -\dfrac{Ap}{C}$ and $\sin\alpha = -\dfrac{Bp}{C}$

Squaring and adding both, we have

$$\cos^2\alpha + \sin^2\alpha = \frac{A^2p^2}{C^2} + \frac{B^2p^2}{C^2}$$

or $\quad 1 = \dfrac{p^2}{C^2}(A^2 + B^2)$

or $\quad p^2 = \dfrac{C^2}{A^2 + B^2}$

or $\quad p = \pm\dfrac{C}{\sqrt{A^2 + B^2}}$

But p is the measure of the perpendicular segment, and is, therefore, taken to be positive.
Assume that $C \geq 0$ without loss of generality.
Hence, $p = \dfrac{C}{\sqrt{A^2 + B^2}}$
Therefore, $\cos\alpha = -\dfrac{A}{\sqrt{A^2 + B^2}}$
and $\sin\alpha = -\dfrac{B}{\sqrt{A^2 + B^2}}$
Hence (5.9) takes the form

$$-\frac{A}{\sqrt{A^2 + B^2}}x - \frac{B}{\sqrt{A^2 + B^2}}y = \frac{C}{\sqrt{A^2 + B^2}}$$

which is the required equation in normal form.

5.9 Angle between Two Lines

We shall consider any two non-perpendicular lines ℓ_1 and ℓ_2, neither of which is parallel to the y-axis and derive a formula for the angle from ℓ_1 to ℓ_2 in terms of their slopes.

The angle between the lines ℓ_1 and ℓ_2 is either acute or obtuse. (See Fig. 5.7) From Fig. 5.7 (i) we see that

$$\alpha_2 = \alpha_1 + \theta,$$

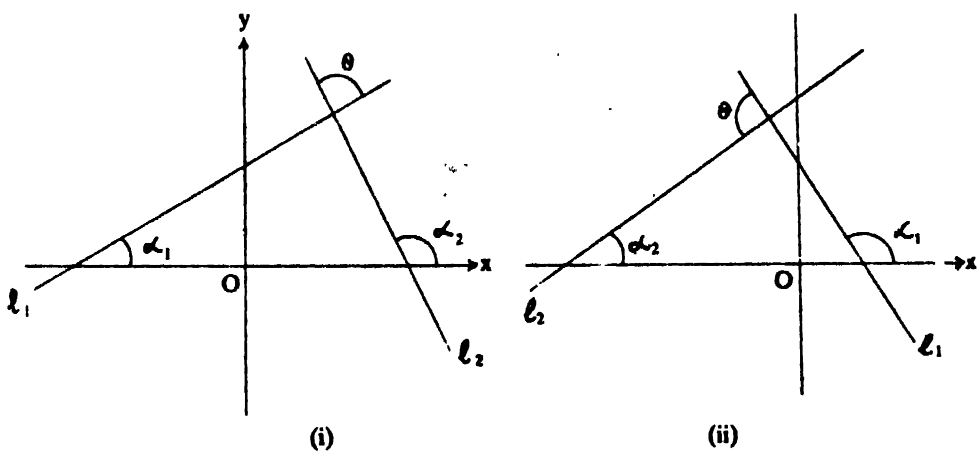


Fig. 5.7

since α_2 is an exterior angle of a triangle with α_1 and θ as the opposite interior angles. Therefore,

$$\theta = \alpha_2 - \alpha_1$$

and $$\tan\theta = \tan(\alpha_2 - \alpha_1)$$

$$= \frac{\tan\alpha_2 - \tan\alpha_2}{1 + \tan\alpha_1 \tan\alpha_2}$$

Therefore, $$\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2} \qquad (5.10)$$

where $m_1 = \tan\alpha_1$, and $m_2 = \tan\alpha_2$

From Fig. 5.7(ii), we see that

$$\alpha_1 = \alpha_2 + (\pi - \theta),$$

since $(\pi - \theta)$ and α_2 are interior angles with α_1 as the opposite exterior angle.

Therefore, $\theta = \alpha_2 - \alpha_1 + \pi$

or $\tan\theta = \tan(\pi + (\alpha_2 - \alpha_1))$

$$= \tan(\alpha_2 - \alpha_1)$$
$$= \frac{\tan\alpha_2 - \tan\alpha_1}{1 + \tan\alpha_2 \tan\alpha_1}$$

Thus $$\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2}$$

We have thus proved the following theorem.

Theorem 5.3

The positive angle θ from the line ℓ_1 to the line ℓ_2 with slopes $m_1 = \tan\alpha_1$ and $m_2 = \tan\alpha_2$ respectively is given by

$$\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2}$$

We note that if two lines are perpendicular to each other, then, as seen earlier,

$$m_1 = -\frac{1}{m_2}, \quad \text{that is} \quad 1 + m_1 m_2 = 0$$

Thus, $\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2}$ is not defined in this case.

Notice that in numerical examples, the value of $\tan\theta$ will sometimes be found to be negative. This would merely mean that instead of getting the acute angle of intersection, its supplement, which too is the angle of intersection of the lines, is being obtained.

Example 5.9

Determine the angle B of the triangle with vertices $A(-2,1), B(2,3)$ and $C(-2,-4)$.

Solution

Let BA be ℓ_1, BC be ℓ_2 so that formula 5.10 will give the desired angle (See Fig. 5.8).

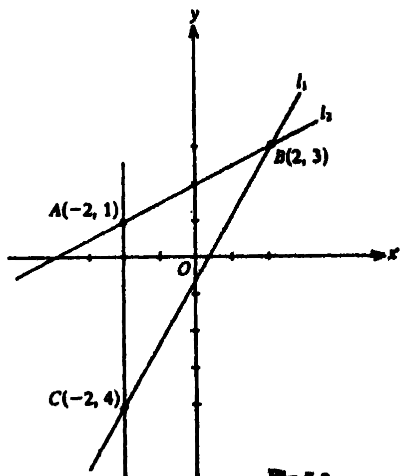


Fig.5.8

Then,
$$m_1 = \frac{1-3}{-2-2} = \frac{1}{2}$$
$$m_2 = \frac{-4-3}{-2-2} = \frac{7}{4}$$

and
$$\tan\angle B = \frac{\frac{7}{4} - \frac{1}{2}}{1 + \frac{7}{4}\cdot\frac{1}{2}} = \frac{\frac{5}{4}}{\frac{15}{8}} = \frac{2}{3} = 0.667$$

Thus $\angle B \simeq 33°42'$.

Example 5.10

Find the value of m_1 if $m_2 = \frac{1}{2}$ and $\theta = \frac{\pi}{4}$.

Solution

From the formula

$$\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2},$$

we get
$$\tan\frac{\pi}{4} = \frac{\frac{1}{2} - m_1}{1 + \frac{m_1}{2}}$$

or
$$1 = \frac{1 - 2m_1}{2 + m_1}$$

or
$$2 + m_1 = 1 - 2m_1$$

Therefore,
$$m_1 = -\frac{1}{3}$$

Condition for Perpendicularity

The equation $Ax + By + C = 0$ may be written in the form

$$y = -\frac{A}{B}x - \frac{C}{B}$$

if $B \neq 0$, from which, by comparison with the form

$$y = mx + b,$$

we see that the slope is $(-\frac{A}{B})$. If $B = 0$, the line is parallel to the y-axis and its slope is not defined.

Let the equation of two lines which are not prallel to the y-axis be

$$A_1x + B_1y + C_1 = 0 \qquad \text{(i)}$$

and
$$A_2x + B_2y + C_2 = 0 \qquad \text{(ii)}$$

In the slope form

$$y = -\frac{A_1}{B_1}x - \frac{C_1}{B_1} \qquad \text{(iii)}$$

and
$$y = -\frac{A_2}{B_2}x - \frac{C_2}{B_2} \qquad \text{(iv)}$$

If m_1 and m_2 are slopes of (iii) and (iv), then $m_1 = \frac{-A_1}{B_1}$ and $m_2 = \frac{-A_2}{B_2}$

We know that the condition for perpendicularity of two lines whose slopes are m_1 and m_2 is $m_1m_2 = -1$

This means
$$\left(\frac{-A_1}{B_1}\right)\left(\frac{-A_2}{B_2.}\right) = -1$$

or $$\frac{A_1A_2}{B_1B_2} = -1$$

or $$A_1A_2 + B_1B_2 = 0 \tag{5.11}$$

which is the condition for perpendicularity of the lines given in the general form.

Note: If $B_1 = 0$, any line parallel to the x-axis will be perpendicular to the line $Ax + By + C = 0$. In this case too (5.11) is satisfied since $A_2 = 0$.

5.10 Condition for Concurrency of Three Straight Lines

Let $$A_1x + B_1y + C_1 = 0 \tag{i}$$

and $$A_2x + B_2y + C_2 = 0. \tag{ii}$$

be the two straight lines AL_1 and AL_2 respectively intersecting each other at the point A as shown in Fig. 5.9.

Since (i) is the equation of AL_1, the coordinates of any point on it must satisfy the equation (i). Similarly, the coordinates of any point on AL_2 must satisfy (ii).

Now, the only point which is common to both the straight lines is their point of intersection A. The coordinates of this point must, therefore, satisfy both (i) and (ii). If (x_1, y_1) are the coordinates of A, then we have

$$A_1x_1 + B_1y_1 + C_1 = 0 \tag{iii}$$

$$A_2x_1 + B_2y_1 + C_2 = 0. \tag{iv}$$

Solving (iii) and (iv), we have

$$\frac{x_1}{B_1C_2 - B_2C_1} = \frac{y_1}{C_1A_2 - A_1C_2} = \frac{1}{A_1B_2 - B_1A_2}$$

This means

$$x_1 = \frac{B_1C_2 - B_2C_1}{A_1B_2 - B_1A_2}$$

and $$y_1 = \frac{C_1A_2 - A_1C_2}{A_1B_2 - B_1A_2}$$

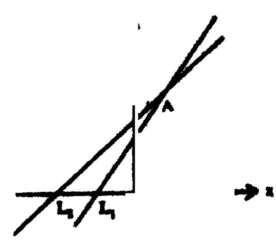


Fig.5.9

Hence, the point of intersection of the two lines (i) and (ii) is

$$\left(\frac{B_1C_2 - B_2C_1}{A_1B_2 - B_1A_2}, \frac{C_1A_2 - A_1C_2}{A_1B_2 - B_1A_2}\right)$$

Now the condition that the three lines whose equations are

$$A_1x + B_1y + C_1 = 0 \tag{v}$$

$$A_2x + B_2y + C_2 = 0 \tag{vi}$$

$$A_3x + B_3y + C_3 = 0 \tag{vii}$$

should be concurrent is that the point of intersection of (v) and (vi) must lie on (vii). In other words, the coordinates of the point of intersection of (v) and (vi) should satisfy the equation of (vii) i.e.

$$A_3\left(\frac{B_1C_2 - B_2C_1}{A_1B_2 - A_2B_1}\right) + B_3\left(\frac{C_1A_2 - C_2A_1}{A_1B_2 - A_2B_1}\right) + C_3 = 0$$

or

$$A_3(B_1C_2 - B_2C_1) + B_3(C_1A_2 - C_2A_1) + C_3(A_1B_2 - A_2B_1) = 0$$

which is the required condition for concurrency.

This condition can be expressed in the determinant form as

$$\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = 0 \tag{5.12}$$

We have proved that if the lines (v), (vi), (vii) are concurrent then (5.12) is true.

Conversely, if (5.12) is true, then it would imply that

$$A_3\left(\frac{B_1C_2 - B_2C_1}{A_1B_2 - A_2B_1}\right) + B_3\left(\frac{C_1A_2 - C_2A_1}{A_1B_2 - A_2B_1}\right) + C_3 = 0$$

i.e. the point

$$\left(\frac{B_1C_2 - B_2C_1}{A_1B_2 - A_2B_1}, \frac{C_1A_2 - C_2A_1}{A_1B_2 - A_2B_1}\right),$$

which, as we have seen, is the point of intersection of (v) and (vi), lies on (vii). Thus if (5.12) is true then (v), (vi), (vii) are concurrent.

Thus (5.12) is the necessary and sufficient condition for the concurrency of (v), (vi) and (vii).

5.11 Analytical Proofs of Geometric Theorems

We can prove many theorems of plane geometry with the help of the formulae of coordinate geometry. We consider some examples.

Example 5.11

Prove that the line segment joining the mid-points of two sides of a triangle is parallel to the third side and equal to one-half its length.

Solution

Consider a triangle OPQ with vertices at $O(0,0)$, $P(a,0)$ and $Q(b,c)$. (See Fig. 5.10) The mid-point A of OQ is $\left(\frac{b}{2}, \frac{c}{2}\right)$. The mid-point B of PQ is $\left(\frac{a+b}{2}, \frac{c}{2}\right)$. The slope m_1 of AB is given by

$$m_1 \quad \frac{\frac{c}{2}-\frac{c}{2}}{\left(\frac{a+b}{2}-\frac{b}{2}\right)}$$

The slope m_2 of OP is given by

$$m_2 = \frac{0-0}{a-0} = 0$$

Thus $m_1 = m_2$.

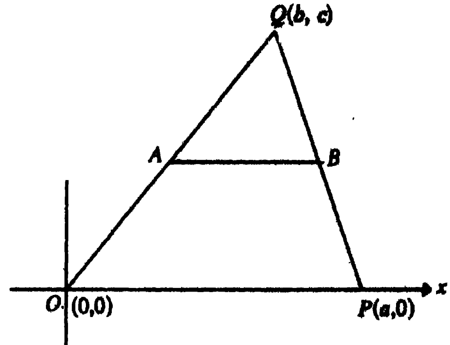


Fig.5.10

Hence, the line joining the mid-points of the two sides is parallel to the third side.

Also, $$AB = \sqrt{\left(\frac{a+b}{2}-\frac{b}{2}\right)^2 + \left(\frac{c}{2}-\frac{c}{2}\right)^2} = \frac{a}{2},$$

and $$OP = \sqrt{(0-a)^2 + (0-0)^2} = a$$

Hence, the line joining the two mid-points A and B is equal to one-half of the base.

Example 5.12
Prove that the diagonals of a parallelogram bisect each other.

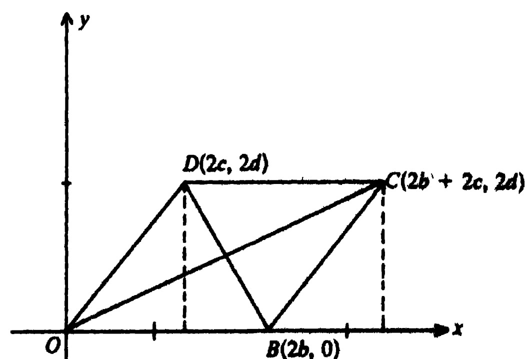


Fig.5.11

Solution

To prove that the diagonals bisect each other, it is only necessary to show that the mid-point of each diagonal is the same point. Let $OBCD$ be the parallelogram with three of its vertices at the points $O(0,0)$, $B(2b,0)$ and $D(2c,2d)$ (See Fig. 5.11). It may be seen that the fourth vertex is $C(2b+c, 2d)$. The mid-point of both the diagonals OC and BD is $(b+c, d)$.

This completes the proof.

Example 5.13

Prove that the segments joining the mid-points of the adjacent sides of a quadrilateral form a parallelogram.

Solution

Let the vertices of the quadrilateral be at the points $O(0,0), A(2a,0), B(2b,2c), C(2d,2e)$ (See Fig. 5.12). Let M_1, M_2, M_3 and M_4 represent the mid-points of OC, CB, BA and OA respectively. To prove that the quadrilateral $M_1M_2M_3M_4$ is a parallelogram, we have only to show that side M_2M_3 is parallel to side M_1M_4, and side M_1M_2 is parallel to M_4M_3.

Coordinates of M_1 are (d, e)
Coordinates of M_2 are $(b+d, c+e)$
Coordinates of M_3 are $(a+b, c)$
Coordinates of M_4 are $(a, 0)$
We now find slopes of
M_1M_2, M_2M_3, M_4M_3 and M_4M_1

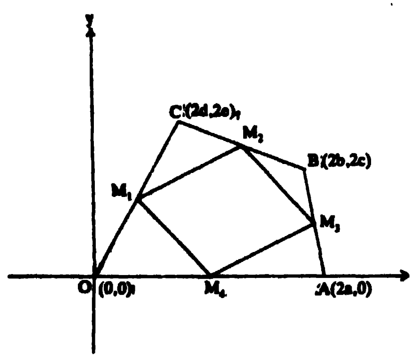


Fig.5.12

$$\text{Slope of } M_1M_2 = \frac{c+e-e}{b+d-d} = \frac{c}{b}$$

$$\text{Slope of } M_2M_3 = \frac{c-(c+e)}{a+b-(b+d)} = \frac{-e}{a-d}$$

$$\text{Slope of } M_4M_3 = \frac{0-c}{a-(a+b)} = \frac{c}{b}$$

$$\text{Slope of } M_4M_1 = \frac{0-e}{a-d} = \frac{-e}{a-d}$$

We see that slope of $M_1M_2 =$ slope of M_4M_3 and slope of $M_2M_3 =$ slope of M_4M_1. Thus, $M_1M_2M_3M_4$ is a parallelogram.

Example 5.14

Prove that the diagonals of the rhombus are perpendicular to each other.

Solution

Let $OABC$ be a rhombus with its vertices as $(0,0), (x_1,0), (x_1+x_2, y_2)$ and (x_2, y_2). As $OABC$ is a rhombus, all of its sides are equal.

Hence, $\quad OA = OC$

or $\quad OA^2 = OC^2$

i.e, $\quad x_1^2 = x_2^2 + y_2^2$

To show that its diagonals are perpendicular to each other, we shall show that the product of the slopes of the diagonals is -1.

Now slope of $OB = \dfrac{y_2}{x_1 + x_2}$

and slope of $AC = \dfrac{y_2}{x_2 - x_1}$

Hence, the product of the slopes

$$= \frac{y_2}{x_1 + x_2} \cdot \frac{y_2}{x_2 - x_1}$$

$$= \frac{y_2^2}{x_2^2 - x_1^2}$$

$$= \frac{y_2^2}{-y_2^2} = -1 \quad \{\text{using (i)}\}$$

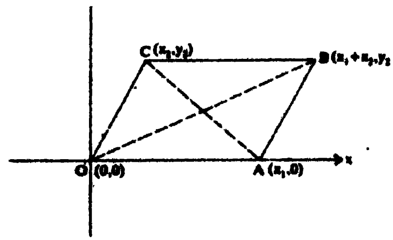


Fig.5.13

Hence, the diagonals of a rhombus are perpendicular to each other.

5.12 Distance of a Point from a Line

Case I: Let the equation of a line AB be

$$x \cos\alpha + y \sin\alpha = p \qquad \text{(i)}$$

Let P be the point (x', y') and d be the length of the perpendicular NP, P being assumed to lie on the opposite side of the line AB from O.

Draw $A'PB'$ parallel to AB through P, and draw the common perpendicular OTT' from O on AB and $A'B'$ (See Fig. 5.14).

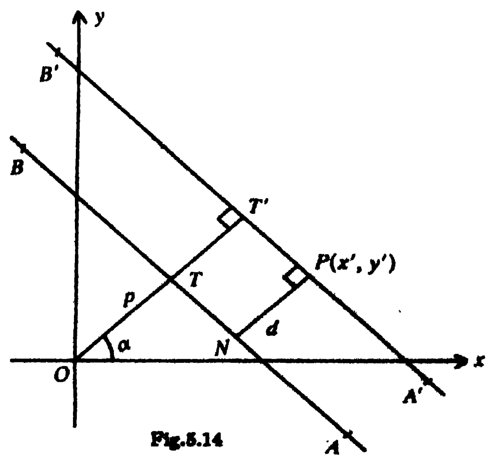


Fig.5.14

Then $OT = p$ and the angle $AOT = \alpha$.

The perpendicular from O on $A'B'$ is of length OT', and makes an angle α with OX; hence the equation of $A'B'$ is

$$x \cos\alpha + y \sin\alpha = p + d$$

The coordinates (x', y') of P satisfy the equation of $A'B'$

$$\therefore \quad x' \cos\alpha + y' \sin\alpha = p + d$$

or

$$d = x' \cos\alpha + y' \sin\alpha - p$$

Hence, the length of the perpendicular is the result of substituting the coordinates of P in the expression $x\cos\alpha + y\sin\alpha - p$.

Case II: Let the equation of the line be given in the general form

$$Ax + By + C = 0 \tag{ii}$$

Reducing the general equation to the normal form, we have

$$\frac{-Ax}{\sqrt{A^2+B^2}} - \frac{By}{\sqrt{A^2+B^2}} = \frac{C}{\sqrt{A^2+B^2}}$$

Now, the length of the perpendicular segment drawn from the given point (x', y') to (ii) is

$$\begin{aligned} &\frac{-Ax' - By' - C}{\sqrt{A^2+B^2}} \\ &= -\frac{Ax' + By' + C}{\sqrt{A^2+B^2}} \end{aligned}$$

Neglecting the negative sign as the length of a segment is always positive, we have the length of the perpendicular segment as

$$\left|\frac{Ax' + By' + C}{\sqrt{A^2+B^2}}\right| \tag{5.13}$$

Example 5.15

Find the distance between the line $3x - 4y + 12 = 0$ and the point (4, 1).

Solution

Since A=3, B=−4, C=12, $x_1 = 4$ and $y_1 = 1$, by the formula for the distance between a line and a point, we have

$$\begin{aligned} d &= \frac{|3 \times 4 + (-4) \times 1 + 12|}{\sqrt{(3)^2 + (-4)^2}} \\ &= \frac{|12 - 4 + 12|}{\sqrt{25}} = \frac{20}{5} = 4 \end{aligned}$$

Example 5.16

Find the perpendicular distance of the point (a, b) from the line $\frac{x}{a} + \frac{y}{b} = 1$.

Solution

The required distance is

$$d = \frac{\frac{a}{a}+\frac{b}{b}-1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}$$

$$\frac{ab}{\sqrt{a^2+b^2}}$$

5.13 Translation of Axes

Let (x, y) be the coordinates of any point P referred to the axes OX and OY. Let $O'X'$ and $O'Y'$ be the new axes parallel to OX and OY. Let O' being the new origin. Let (h, k) be the coordinates of O' referred to the old axes $|OM$ and MP are the abscissa and ordinate of the point P referred to the old axes i.e. $OM = x$ and $MP = y$. $O'M' = x'$ and $M'P$ are the abscissa and ordinate of P referred to the new axes $O'X'$ and $O'Y'$ Let $O'M' = x'$ and $M'P = y'$. If we want to transform an equation in x and y into corresponding equation in x' and y', we will have to express the old coordinates x, y in terms of the new ones x', y' so that the transformation can be performed by direct substitution. It is easily seen from the figure that

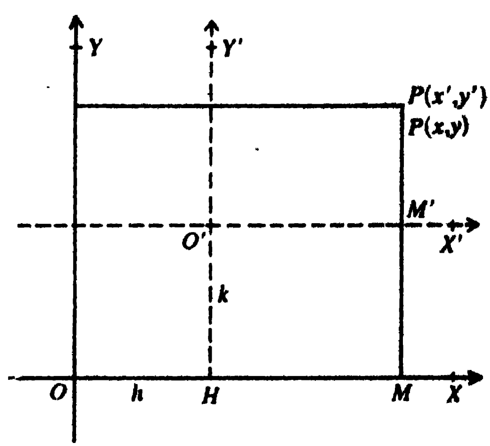


Fig.5.15

$$OM = h + O'M'; \; MP = k + M'P,$$

or

$$x = x' + h, \quad y = y' + k \tag{5.14}$$

We have, therefore, to write $(x' + h)$ for x and $(y' + k)$ for y in the equation which we wish to transform. We thus get an equation in x', y'. So if the equation of the set of points P (locus of P) with respect to OX and OY be $f(x, y) = 0$, the equation to the same set of points when the origin is transferred to O', the axes retaining their directions, becomes $f(x' + h, y' + k) = 0$ where x', y' are the current coordinates with reference to the new axes.

We can easily check that the area of a triangle (or a quadrilateral) and a slope of a line remain unaffected with the change of axes, i.e. the area of a triangle and slope of a line are invariant under the translation of axes.

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of a triangle ABC referred to some rectangular coordinate axes. Let the origin be shifted to the point (h, k) with axes retaining their directions. Then the area Δ of the triangle ABC with respect to the old coordinate axes is given by

$$\Delta = \frac{1}{2}\Big[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\Big] \qquad \text{(i)}$$

Let (x_1', y_1'), (x_2', y_2') and (x_3', y_3') be the coordinates of A, B, C, with respect to the new axes. Then we know that

$$\begin{aligned} x_1 &= x_1' + h, \ \ y_1 = y_1' + k \\ x_2 &= x_2' + h, \ \ y_2 = y_2' + k \\ x_3 &= x_3' + h, \ \ y_3 = y_3' + k \end{aligned}$$

Now substituting these values in (i), we get

$$\begin{aligned} \Delta &= \frac{1}{2}\Big[(x_1' + h)(y_2' + k - y_3' - k) + (x_2' + h)(y_3' + k - y_1' - k) + (x_3' + h) \times \\ &\qquad (y_1' + k - y_2' - k)\Big] \\ &= \frac{1}{2}\Big[x_1'(y_2' - y_3') + x_2'(y_3' - y_1') + x_3'(y_1' - y_2')\Big] \end{aligned}$$

which is nothing but the area of the triangle with respect to the new coordinate axes. Thus the area with reference to both the coordinate axes remains the same.

Similarly, we can prove that the slope of a line does not change with the change of axes (parallel translation).

For, if

$$Ax + By + C = 0 \qquad \text{(ii)}$$

is the equation of a straight line referred to the old coordinate axes, then the slope of the line is seen to be equal to

$$m = -\frac{A}{B}$$

Now, if the origin is shifted to the point (h, k), then as before any point (x', y') on the straight line with respect to the new coordinate axes will satisfy the following relations.

$$x = x' + h \text{ and } y = y' + k$$

Therefore, the equation of the straight line can be written as

$$\begin{aligned} A(x' + h) + B(y' + k) + C &= 0 \\ \text{or} \qquad Ax' + By' + (Ah + Bk + C) &= 0 \qquad \text{(iii)} \end{aligned}$$

which is the equation of the straight line in the new coordinate axes. Now, we can easily see that the slope of the straight line given by (iii) is

$$m' = -\frac{A}{B}$$

which is the same as m. Therefore, we have established that the slope of a straight line is invariant under the translation of axes.

In fact, as we have learned earlier, all the geometric properties in Euclidean Geometry remain unchanged under rigid motion. Since parallel translation of axes is only a particular case of rigid motion (which includes rotation also), it is quite obvious that all geometric properties shall remain unchanged when we transform the coordinates in this way.

We shall see later that translation of axes provides a very useful tool for obtaining equations of different loci in simple form, or for providing simple proofs of geometric properties.

EXERCISE 5.1

Find the equation of the line in each of the problems 1 to 4.

1. Through (4, 3) and slope 2.
2. Through (0,–2) with slope –4.
3. Through (0, –3) and (5, 0).
4. Through (–1, –2) and (–5, –2).
5. Find the lines through the point (0, 2) making an angle $\frac{\pi}{2}$ and $\frac{2\pi}{3}$ with x-axis. Also find the lines parallel to them cutting the y-axis at a distance 2 below the origin. Find their point of intersection with x-axis.
6. What are the inclinations to the x-axis of the lines
$$y = \frac{1}{3}x\sqrt{3} + 3 \text{ and } y = \sqrt{3}x + 3?$$
Show that the line $y = x + 3$ bisects the angle between them.
7. Find the equation of the line that has y-intercept 4 and is parallel to the line $2x-3y = 7$.
8. Find the equation of the line that has x-intercept –3 and is perpendicular to the line $3x + 5y = 4$.
9. Find the equation of a straight line passing through the point (2, 2), such that the sum of its intercepts on the axes is 9.
10. The vertices of a triangle are the points (0, 0), (2, 4) and (6, 4). Find the equations of its sides.

11. The mid-points of the sides of a triangle are (2, 1), (–5, 7), (–5, –5). Find the equations of the sides.

12. Find the equation of the line that is parallel to $2x + 5y = 7$ and passes through the mid-point of the line joining (2, 7) and (–4, 1).

13. Find the equation of the line that is perpendicular to $3x + 2y = 8$ and passes through the mid-point of the line joining (5, –2) and (2, 2).

14. Find the equation of the straight line bisecting the segment joining the points (5, 3) and (4, 4) and making an angle of 45° with the x-axis.

15. If p be the measure of the perpendicular segment from the origin on the line whose intercepts on the axes are a and b, show that

$$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$$

16. Obtain the perpendicular form of the equation of lines from the given values of p and ω,

(i) $p = 3,\ \omega = 45^\circ$ (ii) $p = 5,\ \omega = 30^\circ$
(iii) $p = 5,\ \omega = 135^\circ$ (iv) $p = 1,\ \omega = 90^\circ$

17. Reduce each of the following to the perpendicular form and find p.

(i) $x + y - 2 = 0$ (ii) $4x + 3y - 9 = 0$
(iii) $x - 4 = 0$ (iv) $y - 2 = 0$

18. Which of the lines $2x - y + 3 = 0$ and $x - 4y - 7 = 0$ is farther from the origin?

19. The perpendicular distance of a line from the origin is 5 cm and its slope is –1. Find the equation of the line.

20. Show that the origin is equidistant from the three straight lines,
$4x + 3y + 10 = 0,\ \ 5x - 12y + 26 = 0,\ \ 7x + 24y = 50$

21. Find the angle between the straight lines $y - \sqrt{3}x - 5 = 0$ and $\sqrt{3}y - x + 6 = 0$.

22. Find the equation of the lines through the origin making angle of 60° with the line $x + y\sqrt{3} + 3\sqrt{3} = 0$, also the coordinates of points where they meet the line.

Classify the pairs of lines in Exercise 23 to 25 as coincident, parallel, or intersecting.

23. $6x + 14y - 16 = 0,\quad 12x + 28y - 32 = 0$

24. $3x - 4y = 8,\quad 3x + 4y = 11$

25. $x - 2y = 7,\quad 4y - 2x = 13$

Find the distance between the line and the point in each of the following Exercises from 26 to 28.

26. $4x + 3y - 5 = 0.\quad (-2, -1)$

27. $5x + 12y - 41 = 0,\quad (3, 0)$

28. $y = 4,\quad (2, 3)$

CHAPTER 6

Family of Lines

6.1 Equation of Family of Lines

Let $$A_1x + B_1y + C_1 = 0 \qquad (6.1)$$
and $$A_2x + B_2y + C_2 = 0 \qquad (6.2)$$

be two given non-parallel lines. Then, for any real number k,

$$(A_1x + B_1y + C_1) + k(A_2x + B_2y + C_2) = 0 \qquad (6.3)$$

for any values of x and y for which (6.1), (6.2) are both true. In other words the point of intersection (x_0, y_0) of (6.1) and (6.2) lies on the straight line whose equation is (6.3). Hence (6.3) represents a line through the point of intersection of (6.1) and (6.2) for every k. Hence, (6.3) represents the *family of lines* passing through the point of intersection of the two lines.

Note: If there exists no point (x_0, y_0) common to the lines (6.1) and (6.2), they are parallel. In this case (6.3) gives the family of lines parallel to them.

Example 6.1

Find the equation of the straight line parallel to the y-axis and drawn through the point of intersection of $x - 7y + 5 = 0$ and $3x + y - 7 = 0$.

Solution

The equation of any straight line through the point of intersection of the given lines is of the form

$$x - 7y + 5 + k(3x + y - 7) = 0$$
or $$(1 + 3k)x + (k - 7)y + 5 - 7k = 0$$

If this line is parallel to y-axis, the coefficient of y is zero, hence $k = 7$, and the equation becomes

$$x - 2 = 0$$

Example 6.2

Find the equation of the line through the intersection of $3x + 4y = 7$ and $x - y + 2 = 0$, and with slope 5.

Solution

The equation of the family of lines passing through the point of intersection of the given lines will be

$$(3x + 4y - 7) + k(x - y + 2) = 0$$

or

$$(3 + k)x + (4 - k)y + (2k - 7) = 0$$

The slope of each member of the family is

$$\frac{k+3}{k-4}$$

But slope for the line is given to be 5. Hence,

$$\frac{k+3}{k-4} = 5$$

or

$$k + 3 = 5k - 20, \text{ i.e., } \quad k = \frac{23}{4}$$

Therefore, the equation of the desired line is

$$(3x + 4y - 7) + \frac{23}{4}(x - y + 2) = 0$$

or

$$4(3x + 4y - 7) + 23(x - y + 2) = 0$$

or

$$35x - 7y + 18 = 0$$

EXERCISE 6.1

1. Find the equation of the line through the point of intersection of $x + 2y = 5$ and $x - 3y = 7$, and passing through the point
   (i) (0,0) (ii) (2,−3)
   (iii) (1,0) (iv) (0,−1)
2. Find the equation of the line through the point of intersection of $5x - 3y = 1$ and $2x + 3y = 23$, and perpendicular to the line whose equation is
   (i) $x - 2y = 3$ (ii) $x = 0$
   (iii) $y = 0$ (iv) $5x - 3y = 1$
3. Find the equation of the line through the intersection of the lines $x + 2y - 3 = 0$ and $4x - y + 7 = 0$ and which is parallel to $5x + 4y - 20 = 0$
4. Find the equation of the line through the intersection of the lines $2x + 3y - 4 = 0$ and $x - 5y + 7 = 0$ that has its x-intercept equal to −4

6.2 Pair of Straight Lines through Origin

We shall show that the homogeneous equation of the second degree

$$ax^2 + 2hxy + by^2 = 0$$

represents a pair of straight lines passing through the origin if $h^2 \geq ab$.
Solving the above equation as a quadratic equation for x, we get

$$x = \frac{-h \pm \sqrt{h^2 - ab}}{a}\, y, \quad (h^2 \geq ab)$$

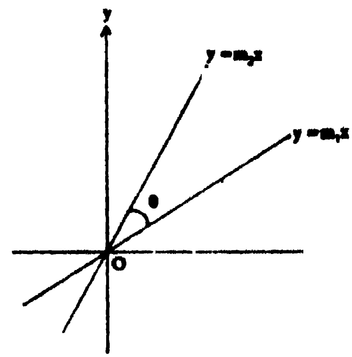


Fig.6.1

i.e., either $\qquad ax + (h + \sqrt{h^2 - ab})y = 0$

or $\qquad ax + (h - \sqrt{h^2 - ab})y = 0.$

Each of the above is a straight line passing through the origin.
Hence, the homogenous equation

$$ax^2 + 2hxy + by^2 = 0$$

represents two straight lines passing through the origin. These lines are real and distinct, if $h^2 > ab$ and coincident if $h^2 = ab$, and the lines do not exist if $h^2 < ab$.

6.3 Angle between the Pair of Straight Lines

Let the pair of straight lines be represented by the equation $ax^2 + 2hxy + by^2 = 0$.

Since the above equati[illegible] its a pair of straight lines passing through the origin, they will be given by the [illegible]

$$y = m_1 x \tag{6.4}$$

and $$y = m_2 x \tag{6.5}$$

or $\qquad y - m_1 x = 0$ and y [illegible] $= 0$

Then $\qquad (y - m_1 x)(y - m_2 x) = 0$

or $\qquad m_1 m_2 x^2 - (m_1 + m_2)xy + y^2 = 0$

This equation is the same as

$$ax^2 + 2hxy + by^2 = 0$$

The equation $ax^2 + 2hxy + by^2 = 0$ is called a homogeneous equation of second degree in x and y as the degree of each of the terms $ax^2, 2hxy, by^2$ is 2 and it does not contain any other term

Therefore comparing the coefficients

$$\frac{m_1 m_2}{a} = -\frac{(m_1+m_2)}{2h} = \frac{1}{b}$$

This gives

$$m_1 m_2 = \frac{a}{b} \tag{6.6}$$

and

$$m_1 + m_2 = -\frac{2h}{b} \tag{6.7}$$

If θ be the angle between the given straight lines, then

$$\begin{aligned}\tan\theta &= \frac{m_2 - m_1}{1 + m_1 m_2}\\ &= \frac{\sqrt{(m_1+m_2)^2 - 4m_1 m_2}}{1 + m_1 m_2}\end{aligned}$$

Substituting the values of $m_1 m_2$ and $m_1 + m_2$, we obtain

$$\tan\theta = \frac{\sqrt{4h^2 - 4ab}}{a+b} \tag{6.8}$$

Hence, $$\theta = \tan^{-1}\left\{\frac{2\sqrt{h^2 - ab}}{a+b}\right\} \tag{6.9}$$

Notice that the condition for coincidence of the lines is

$$h^2 = ab \tag{6.10}$$

In other words, for the coincidence of lines, the expression $ax^2 + 2hxy + by^2$ should be a perfect square.

The condition for perpendicularity is

$$a + b = 0 \tag{6.11}$$

6.4 Equations of the Bisectors of the Angles

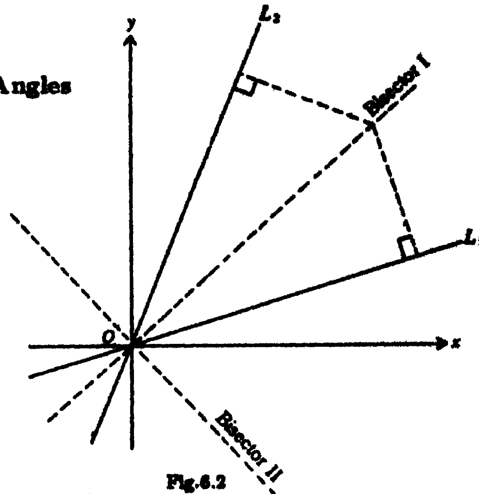


Fig.6.2

Let the lines be represented by the equation

$$ax^2 + 2hxy + by^2 = 0$$

Suppose the lines are

$$y - m_1x = 0, \quad y - m_2x = 0$$

Since from any point on a bisector the perpendicular distances on the lines are equal, we have

$$\frac{y - m_1x}{\sqrt{1 + m_1^2}} = \pm\frac{y - m_2x}{\sqrt{1 + m_2^2}}$$

The two bisectors can be expressed in one equation which is

$$\left(\frac{y - m_1x}{\sqrt{1+m_1^2}} + \frac{y - m_2x}{\sqrt{1+m_2^2}}\right)\left(\frac{y - m_1x}{\sqrt{1+m_1^2}} - \frac{y - m_2x}{\sqrt{1+m_2^2}}\right) = 0$$

or $$(1 + m_2^2)(y - m_1x)^2 - (1 + m_1^2)(y - m_2x)^2 = 0$$

or $$x^2(m_1^2 - m_2^2) - 2xy(m_1 - m_2)(1 - m_1m_2) + y^2(m_2^2 - m_1^2) = 0$$

or $$x^2 - y^2 = 2xy\frac{1 - m_1m_2}{m_1 + m_2} \quad (\text{since } m_1 - m_2 \neq 0)$$

Substituting $\frac{a}{b}$ for m_1m_2 and $-\frac{2h}{b}$ for $m_1 + m_2$ [see (6.6) and (6.7)], the above equation becomes

$$x^2 - y^2 = 2xy\frac{a - b}{2h} \tag{6.12}$$

i.e. $$\frac{x^2 - y^2}{a - b} = \frac{xy}{h} \tag{6.13}$$

which is the required equation of the bisectors of the angles between the pair of lines given by $ax^2 + 2hxy + by^2 = 0$.

Example 6.3

Find the equation of the lines bisecting the angles between the pair of lines $3x^2 + xy - 2y^2 = 0$.

Solution

Using the formula $\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$, we get the equation

$$\frac{x^2 - y^2}{3 - (-2)} = \frac{xy}{\frac{1}{2}}$$

or $$x^2 - 10xy - y^2 = 0$$

EXERCISE 6.2

1. Find separate equations of the straight lines whose joint equation is

$$x^2 - 5xy + 6y^2 = 0$$

2. Find the separate equation of the straight lines whose joint equation is

$$ab(x^2 - y^2) + (a^2 - b^2)xy = 0$$

3. Find the angle between the lines whose joint equation is $2x^2 - 3xy - 6y^2 = 0$.
4. Find the straight lines represented by the equation

$$y^2 - xy - 6x^2 = 0$$

and find the angle between them.

5. Prove that the angle between the straight lines given by

$$(x\cos\alpha - y\sin\alpha)^2 = (x^2 + y^2)\sin^2\alpha$$

is 2α.

6. Show that the bisectors of the angles between the lines

$$(ax + by)^2 = c(bx - ay)^2$$

are respectively parallel and perpendicular to the line $ax + by + c = 0$.

6.5 Condition for the General Equation of Second Degree to Represent Two Straight Lines

We have seen that the general equation of the first degree in x and y, viz., $Ax+By+C = 0$ represents a straight line. Let us now consider the product equation.

$$(Ax + By + C)(A'x + B'y + C') = 0 \tag{6.14}$$

and examine what locus is represented by this.

Since the two factors on the left hand side of equation (6.14) are $Ax + By + C$ and $A'x + B'y + C'$, the equation is obviously satisfied if either of these factors is equal to zero. Hence, equation (6.14) represents a pair of straight lines

$$Ax + By + C = 0 \quad \text{and} \quad A'x + B'y + C' = 0.$$

If we multiply the two factors on the left hand side of (6.14), we get an equation of the form

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \tag{6.15}$$

where the values of a, b, c, f, g, h are easily determined in terms of A, B, C, A', B', C'. This is the most general equation of the second degree and will represent a pair of straight lines provided the coefficients of various powers of x and y and the constant term are suitably determined. The obvious condition is that the expression on the left hand side of equation (6.15) should break up into linear factors in x and y.

Let the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

represent a pair of straight lines and let (x_1, y_1) be the point of intersection of the lines.

If the origin is transferred to (x_1, y_1) the axes remaining parallel to their original directions, equation (6.15) transforms into

$$a(x + x_1)^2 + 2h(x + x_1)(y + y_1) + b(y + y_1)^2 + 2g(x + x_1) + 2f(y + y_1) + c = 0 \tag{6.16}$$

Referred to new axes (6.16) is the equation of two straight lines through origin, and must therefore be a homogeneous quadratic equation in x and y. Simplifying further, we have

$$\begin{aligned} ax^2 + 2hxy + by^2 + 2(ax_1 + hy_1 + g)x + 2(hx_1 + by_1 + f)y + ax_1^2 \\ +2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = 0 \end{aligned} \tag{6.17}$$

Hence, (6.17) is a homogeneous quadratic equation in x and y with respect to new coordinate axes. The pair of straight lines passes through origin. Hence, the coefficients of x and y and the constant in (6.17) must separately vanish.

Thus
$$ax_1 + hy_1 + g = 0 \tag{6.18}$$
$$hx_1 + by_1 + f = 0 \tag{6.19}$$
and
$$ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = 0 \tag{6.20}$$

Multiplying (6.18) by x_1, (6.19) by y_1, adding and subtracting the result from (6.20), we get

$$gx_1 + fy_1 + c = 0 \tag{6.21}$$

If we solve (6.18) and (6.19), we find that

$$x_1 = \frac{fh - gb}{h^2 - ab}, \qquad y_1 = \frac{fa - gh}{h^2 - ab}$$

If we put these values in (6.21), and simplify, we get

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

This can be expressed in the determinant form as

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

which is the condition that the equation (6.15) should represent a pair of straight lines.

Note: The point of intersection can be found by solving any two of the equations (6.18), (6.19) and (6.21).

We can arrive at the same result by the following alternative method. The equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

represents a pair of straight lines if the expression on the left can be broken up into two linear factors of the type

$$lx + my + n \qquad \text{and} \qquad l'x + m'y + n'.$$

We then have

$$\begin{aligned} &ax^2 + 2hxy + by^2 + 2gx + 2fy + c \\ &\equiv (lx + my + n)(l'x + m'y + n') \end{aligned}$$

which gives $ll' = a$, $mm' = b$, $nn' = c$, $lm' + ml' = 2h$, $ln' + nl' = 2g$, $mn' + nm' = 2f$.

Multiplying the last three results together, we obtain

$$2ll'mm'nn' + ll'(m^2n'^2 + n^2m'^2) + mm'(n^2l^2 + l^2n'^2) + nn'(l^2m'^2 + m^2l'^2) = 8fgh$$

which reduces to

$$2abc + a(4f^2 - 2bc) + b(4g^2 - 2ac) + c(4h^2 - 2ab) = 8fgh$$

or

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

Remark

We have noted in the first proof above that if the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of lines, and if we translate the axes so that the point of intersection is taken as new origin, then this equation reduces to the homogeneous form

$$ax^2 + 2hxy + by^2 = 0$$

in new coordinates. We shall be using this fact quite often in what follows.

Sufficiency of the Condition

We have seen that if the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \tag{6.22}$$

represents two straight lines, then

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \tag{6.23}$$

This shows that (6.23) is the necessary condition that the general equation should represent two straight lines. We shall now show that this condition is sufficient also.

If $\quad abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ i.e., $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$

then this is precisely the condition that the lines

$$\begin{aligned} ax + hy + g &= 0 \\ hx + by + f &= 0 \\ gx + fy + c &= 0 \end{aligned}$$

are concurrent. If these lines are concurrent at the point (x_1, y_1), then the equations (6.18), (6.19) and (6.21) are true. If we multiply (6.18) by x_1, (6.19) by y_1 and add them to (6.21), we get (6.20), so (6.20) is also true. And these indicate that when the origin is transferred to (x_1, y_1), the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

becomes

$$ax^2 + 2hxy + by^2 = 0 \tag{6.24}$$

[See (6.17)]. But we know that (6.24) represents a pair of lines through the (new) origin and so if

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

then $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of lines through (x_1, y_1).

6.6 Angle between Two Lines

If the origin is transferred to the point of intersection of the lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ the equation reduces to

$$ax^2 + 2hxy + by^2 = 0$$

The lines $ax^2+2hxy+by^2+2gx+2fy+c=0$ are, therefore, parallel to the pair of lines

$$ax^2+2hxy+by^2=0$$

through the origin.

The angle between the given lines is, therefore,

$$\tan^{-1}\left(\frac{2\sqrt{h^2-ab}}{a+b}\right)$$

The lines are parallel if $h^2=ab$, and perpendicular if $a+b=0$.

EXERCISE 6.3

1. Find what the following equations become when the origin is shifted to the point (1,1)

   (i) $x^2+xy-3x-y+2=0$

   (ii) $xy-y^2-x+y=0$

   (iii) $xy-x-y+1=0$

   (iv) $x^2-y^2-2x+2y=0$

2. Show that the equation $3x^2+7xy+2y^2+5x+5y+2=0$ represents a pair of straight lines.

3. Show that the equation $2x^2-5xy+2y^2-3x+3y+1=0$ represents two straight lines intersecting at an angle θ such that $\tan\theta=\frac{3}{4}$

4. Show that the equation $x^2-y^2-x+3y-2=0$ represents a pair of straight lines. Find them, and show they are at right angles.

5. If the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents a pair of straight lines, show that they intersect in the point

$$\left(\frac{hf-bg}{ab-h^2},\ \frac{hg-af}{ab-h^2}\right)$$

6. Show that the four lines given by the equations $3x^2+8xy-3y^2=0$ and $3x^2+8xy-3y^2+2x-4y-1=0$ form a square. Find the equations of the diagonals of the square.

TABLES

TABLE I

Four-Place Values of Trigonometric Functions
Angle θ in Degrees and Radians

Angle θ									
Degrees	Radians	sin θ	csc θ	tan θ	cot θ	sec θ	cos θ		
0° 00′	.0000	.0000	No value	.0000	No value	1.000	1.0000	1.5708	90° 00′
10	029	029	343.8	029	343.8	000	000	679	50
20	058	058	171.9	058	171.9	000	000	650	40
30	.0087	.0087	114.6	.0087	114.6	1.000	1.0000	1.5621	30
40	116	116	85.95	116	85.94	000	.9999	592	20
50	145	145	68.76	145	68.75	000	999	563	10
1° 00′	.0175	.0175	57.30	.0175	57.29	1.000	.9998	1.5533	89° 00′
10	204	204	49.11	204	49.10	000	998	504	50
20	233	233	42.98	233	42.96	000	997	475	40
30	.0262	.0262	38.20	.0262	38.19	1.000	.9997	1.5446	30
40	291	291	34.38	291	34.37	000	996	417	20
50	320	320	31.26	320	31.24	001	995	388	10
2° 00′	.0349	.0349	28.65	.0349	28.64	1.001	.9994	1.5359	88° 00′
10	378	378	26.45	378	26.43	001	993	330	50
20	407	407	24.56	407	24.54	001	992	301	40
30	.0436	.0436	22.93	.0437	22.90	1.001	.9990	1.5272	30
40	465	465	21.49	466	21.47	001	989	243	20
50	495	494	20.23	495	20.21	001	988	213	10
3° 00′	.0524	.0523	19.11	.0524	19.08	1.001	.9986	1.5184	87° 00′
10	553	552	18.10	553	18.07	002	985	155	50
20	582	581	17.20	582	17.17	002	983	126	40
30	.0611	.0610	16.38	.0612	16.35	1.002	.9981	1.5097	30
40	640	640	15.64	641	15.60	002	980	068	20
50	669	669	14.96	670	14.92	002	978	039	10
4° 00′	.0698	.0698	14.34	.0699	14.30	1.002	.9976	1.5010	86° 00′
10	727	727	13.76	729	13.73	003	974	981	50
20	756	756	13.23	758	13.20	003	971	952	40
30	.0785	.0785	12.75	.0787	12.71	1.003	.9969	1.4923	30
40	814	814	12.29	816	12.25	003	967	893	20
50	844	843	11.87	846	11.83	004	964	864	10
5° 00′	.0873	.0872	11.47	.0875	11.43	1.004	.9962	1.4835	85° 00′
10	902	901	11.10	904	11.06	004	959	806	50
20	931	929	10.76	934	10.71	004	957	777	40
30	.0960	.0958	10.43	.0963	10.39	1.005	.9954	1.4748	30
40	989	987	10.13	992	10.08	005	951	719	20
50	.1018	.1016	9.839	.1022	9.788	005	948	690	10
6° 00′	.1047	.1045	9.567	.1051	9.514	1.006	.9945	1.4661	84° 00′
		cos θ	sec θ	cot θ	tan θ	csc θ	sin θ	Radians	Degrees
								Angle θ	

TABLE I—*continued*

Angle θ									
Degrees	Radians	sin θ	csc θ	tan θ	cot θ	sec θ	cos θ		
6° 00′	.1047	.1045	9.567	.1051	9.514	1.006	.9945	1.4661	84° 00′
10	076	074	9.309	080	9.255	006	942	632	50
20	105	103	9.065	110	9.010	006	939	603	40
30	.1134	.1132	8.834	.1139	8.777	1.006	.9936	1.4573	30
40	164	161	8.614	169	8.556	007	932	544	20
50	193	190	8.405	198	8.345	007	929	515	10
7° 00′	.1222	.1219	8.206	.1228	8.144	1.008	.9925	1.4486	83° 00′
10	251	248	8.016	257	7.953	008	922	457	50
20	280	276	7.834	287	7.770	008	918	428	40
30	.1309	.1305	7.661	.1317	7.596	1.009	.9914	1.4399	30
40	338	334	7.496	346	7.429	009	911	370	20
50	367	363	7.337	376	7.269	009	907	341	10
8° 00′	.1396	.1392	7.185	.1405	7.115	1.010	.9903	1.4312	82° 00′
10	425	421	7.040	435	6.968	010	899	283	50
20	454	449	6.900	465	6.827	011	894	254	40
30	.1484	.1478	6.765	.1495	6.691	1.011	.9890	1.4224	30
40	513	507	6.636	524	6.561	012	886	195	20
50	542	536	6.512	554	6.435	012	881	166	10
9° 00′	.1571	.1564	6.392	.1584	6.314	1.012	.9877	1.4137	81° 00′
10	600	593	277	614	197	013	872	108	50
20	629	622	166	644	084	013	868	079	40
30	.1658	.1650	6.059	.1673	5.976	1.014	.9863	1.4050	30
40	687	679	5.955	703	871	014	858	1.4021	20
50	716	708	855	733	769	015	853	992	10
10° 00′	.1745	.1736	5.759	.1763	5.671	1.015	.9848	1.3963	80° 00′
10	774	765	665	793	576	016	843	934	50
20	804	794	575	823	485	016	838	904	40
30	.1833	.1822	5.487	.1853	5.396	1.017	.9833	1.3875	30
40	862	851	403	883	309	018	827	846	20
50	891	880	320	914	226	018	822	817	10
11° 00′	.1920	.1908	5.241	.1944	5.145	1.019	.9816	1.3788	79° 00′
10	949	937	164	974	066	019	811	759	50
20	978	965	089	.2004	4.989	020	805	730	40
30	.2007	.1994	5.016	.2035	4.915	1.020	.9799	1.3701	30
40	036	.2022	4.945	065	843	021	793	672	20
50	065	051	876	095	773	022	787	643	10
12° 00′	.2094	.2079	4.810	.2126	4.705	1.022	.9781	1.3614	78° 00′
10	123	108	745	156	638	023	775	584	50
20	153	136	682	186	574	024	769	555	40
30	.2182	.2164	4.620	.2217	4.511	1.024	.9763	1.3526	30
40	211	193	560	247	449	025	757	497	20
50	240	221	502	278	390	026	750	468	10
13° 00′	.2269	.2250	4.445	.2309	4.331	1.026	.9744	1.3439	77° 00′
		cos θ	sec θ	cot θ	tan θ	csc θ	sin θ	Radians	Degrees
								Angle θ	

TABLE I—*continued*

Angle θ									
Degrees	**Radians**	**sin θ**	**csc θ**	**tan θ**	**cot θ**	**sec θ**	**cos θ**		
13° 00′	.2269	.2250	4.445	.2309	4.331	1.026	.9744	1.3439	**77° 00′**
10	298	278	390	339	275	027	737	410	50
20	327	306	336	370	219	028	730	381	40
30	.2356	.2334	4.284	.2401	4.165	1.028	.9724	1.3352	30
40	385	363	232	432	113	029	717	323	20
50	414	391	182	462	061	030	710	294	10
14° 00′	.2443	.2419	4.134	.2493	4.011	1.031	.9703	1.3265	**76° 00′**
10	473	447	086	524	3.962	031	696	235	50
20	502	476	039	555	914	032	689	206	40
30	.2531	.2504	3.994	.2586	3.867	1.033	.9681	1.3177	30
40	560	532	950	617	821	034	674	148	20
50	589	560	906	648	776	034	667	119	10
15° 00′	.2618	.2588	3.864	.2679	3.732	1.035	.9659	1.3090	**75° 00′**
10	647	616	822	711	689	036	652	061	50
20	676	644	782	742	647	037	644	032	40
30	.2705	.2672	3.742	.2773	3.606	1.038	.9636	1.3003	30
40	734	700	703	805	566	039	628	974	20
50	763	728	665	836	526	039	621	945	10
16° 00′	.2793	.2756	3.628	.2867	3.487	1.040	.9613	1.2915	**74° 00′**
10	822	784	592	899	450	041	605	886	50
20	851	812	556	931	412	042	596	857	40
30	.2880	.2840	3.521	.2962	3.376	1.043	.9588	1.2828	30
40	909	868	487	994	340	044	580	799	20
50	938	896	453	.3026	305	045	572	770	10
17° 00′	.2967	.2924	3.420	.3057	3.271	1.046	.9563	1.2741	**73° 00′**
10	996	952	388	089	237	047	555	712	50
20	.3025	979	357	121	204	048	546	683	40
30	.3054	.3007	3.326	.3153	3.172	1.048	.9537	1.2654	30
40	083	035	295	185	140	049	528	625	20
50	113	062	265	217	108	050	520	595	10
18° 00′	.3142	.3090	3.236	.3249	3.078	1.051	.9511	1.2566	**72° 00′**
10	171	118	207	281	047	052	502	537	50
20	200	145	179	314	018	053	492	508	40
30	.3229	.3173	3.152	.3346	2.989	1.054	.9483	1.2479	30
40	258	201	124	378	960	056	474	450	20
50	287	228	098	411	932	057	465	421	10
19° 00′	.3316	.3256	3.072	.3443	2.904	1.058	.9455	1.2392	**71° 00′**
10	345	283	046	476	877	059	446	363	50
20	374	311	021	508	850	060	436	334	40
30	.3403	.3338	2.996	.3541	2.824	1.061	.9426	1.2305	30
40	432	365	971	574	798	062	417	275	20
50	462	393	947	607	773	063	407	246	10
20° 00′	.3491	.3420	2.924	.3640	2.747	1.064	.9397	1.2217	**70° 00′**
		cos θ	**sec θ**	**cot θ**	**tan θ**	**csc θ**	**sin θ**	**Radians**	**Degrees**
								Angle θ	

TABLE I—*continued*

Angle θ									
Degrees	Radians	sin θ	csc θ	tan θ	cot θ	sec θ	cos θ		
20° 00′	.3491	.3420	2.924	.3640	2.747	1.064	.9397	1.2217	70° 00′
10	520	448	901	673	723	065	387	188	50
20	549	475	878	706	699	066	377	159	40
30	.3578	.3502	2.855	.3739	2.675	1.068	.9367	1.2130	30
40	607	529	833	772	651	069	356	101	20
50	636	557	812	805	628	070	346	072	10
21° 00′	.3665	.3584	2.790	.3839	2.605	1.071	.9336	1.2043	69° 00′
10	694	611	769	872	583	072	325	1.2014	50
20	723	638	749	906	560	074	315	985	40
30	.3752	.3665	2.729	.3939	2.539	1.075	.9304	1.1956	30
40	782	692	709	973	517	076	293	926	20
50	811	719	689	.4006	496	077	283	897	10
22° 00′	.3840	.3746	2.669	.4040	2.475	1.079	.9272	1.1868	68° 00′
10	869	773	650	074	455	080	261	839	50
20	898	800	632	108	434	081	250	810	40
30	.3927	.3827	2.613	.4142	2.414	1.082	.9239	1.1781	30
40	956	854	595	176	394	084	228	752	20
50	985	881	577	210	375	085	216	723	10
23° 00′	.4014	.3907	2.559	.4245	2.356	1.086	.9205	1.1694	67° 00′
10	043	934	542	279	337	088	194	665	50
20	072	961	525	314	318	089	182	636	40
30	.4102	.3987	2.508	.4348	2 300	1.090	.9171	1.1606	30
40	131	.4014	491	383	282	092	159	577	20
50	160	041	475	417	264	093	147	548	10
24° 00′	.4189	.4067	2.459	.4452	2.246	1.095	.9135	1.1519	66° 00′
10	218	094	443	487	229	096	124	490	50
20	247	120	427	522	211	097	112	461	40
30	.4276	.4147	2.411	.4557	2.194	1.099	.9100	1.1432	30
40	305	173	396	592	177	100	088	403	20
50	334	200	381	628	161	102	075	374	10
25° 00′	.4363	.4226	2.366	.4663	2.145	1.103	.9063	1.1345	65° 00′
10	392	253	352	699	128	105	051	316	50
20	422	279	337	734	112	106	038	286	40
30	.4451	.4305	2.323	.4770	2.097	1.108	.9026	1.1257	30
40	480	331	309	806	081	109	013	228	20
50	509	358	295	841	066	111	001	199	10
26° 00′	.4538	.4384	2.281	.4877	2.050	1.113	.8988	1.1170	64° 00′
10	567	410	268	913	035	114	975	141	50
20	596	436	254	950	020	116	962	112	40
30	.4625	.4462	2.241	.4986	2.006	1.117	.8949	1.1083	30
40	654	488	228	.5022	1.991	119	936	054	20
50	683	514	215	059	977	121	923	1.1025	10
27° 00′	.4712	.4540	2.203	.5095	1.963	1.122	.8910	1.0996	63° 00′
		cos θ	sec θ	cot θ	tan θ	csc θ	sin θ	Radians	Degrees
								Angle θ	

TABLE I—*continued*

Angle θ									
Degrees	Radians	sin θ	csc θ	tan θ	cot θ	sec θ	cos θ		
27° 00′	.4712	.4540	2.203	.5095	1.963	1.122	.8910	1.0996	63° 00′
10	741	566	190	132	949	124	897	966	50
20	771	592	178	169	935	126	884	937	40
30	.4800	.4617	2.166	.5206	1.921	1.127	.8870	1.0908	30
40	829	643	154	243	907	129	857	879	20
50	858	669	142	280	894	131	843	850	10
28° 00′	.4887	.4695	2.130	.5317	1.881	1.133	.8829	1.0821	62° 00′
10	916	720	118	354	868	134	816	792	50
20	945	746	107	392	855	136	802	763	40
30	.4974	.4772	2.096	.5430	1.842	1.138	.8788	1.0734	30
40	.5003	797	085	467	829	140	774	705	20
50	032	823	074	505	816	142	760	676	10
29° 00′	.5061	.4848	2.063	.5543	1.804	1.143	.8746	1.0647	61° 00′
10	091	874	052	581	792	145	732	617	50
20	120	899	041	619	780	147	718	588	40
30	.5149	.4924	2.031	.5658	1.767	1.149	.8704	1.0559	30
40	178	950	020	696	756	151	689	530	20
50	207	975	010	735	744	153	675	501	10
30° 00′	.5236	.5000	2.000	.5774	1.732	1.155	.8660	1.0472	60° 00′
10	265	025	1.990	812	720	157	646	443	50
20	294	050	980	851	709	159	631	414	40
30	.5323	.5075	1.970	.5890	1.698	1.161	.8616	1.0385	30
40	352	100	961	930	686	163	601	356	20
50	381	125	951	969	675	165	587	327	10
31° 00′	.5411	.5150	1.942	.6009	1.664	1.167	.8572	1.0297	59° 00′
10	440	175	932	048	653	169	557	268	50
20	469	200	923	088	643	171	542	239	40
30	.5498	.5225	1.914	.6128	1.632	1.173	.8526	1.0210	30
40	527	250	905	168	621	175	511	181	20
50	556	275	896	208	611	177	496	152	10
32° 00′	.5585	.5299	1.887	.6249	1.600	1.179	.8480	1.0123	58° 00′
10	614	324	878	289	590	181	465	094	50
20	643	348	870	330	580	184	450	065	40
30	.5672	.5373	1.861	.6371	1.570	1.186	.8434	1.0036	30
40	701	398	853	412	560	188	418	1.0007	20
50	730	422	844	453	550	190	403	977	10
33° 00′	.5760	.5446	1.836	.6494	1.540	1.192	.8387	.9948	57° 00′
10	789	471	828	536	530	195	371	919	50
20	818	495	820	577	520	197	355	890	40
30	.5847	.5519	1.812	.6619	1.511	1.199	.8339	.9861	30
40	876	544	804	661	501	202	323	832	20
50	905	568	796	703	1.492	204	307	803	10
34° 00′	.5934	.5592	1.788	.6745	1.483	1.206	.8290	.9774	56° 00′
		cos θ	sec θ	cot θ	tan θ	csc θ	sin θ	Radians	Degrees
								Angle θ	

TABLE 1—continued

Angle θ									
Degrees	Radians	sin θ	csc θ	tan θ	cot θ	sec θ	cos θ		
34° 00′	.5934	.5592	1.788	.6745	1.483	1.206	.8290	.9774	56° 00′
10	963	616	781	787	473	209	274	745	50
20	992	640	773	830	464	211	258	716	40
30	.6021	.5664	1.766	.6873	1.455	1.213	.8241	.9687	30
40	050	688	758	916	446	216	225	657	20
50	080	712	751	959	437	218	208	628	10
35° 00′	.6109	.5736	1.743	.7002	1.428	1.221	.8192	.9599	55° 00′
10	138	760	736	046	419	223	175	570	50
20	167	783	729	089	411	226	158	541	40
30	.6196	.5807	1.722	.7133	1.402	1.228	.8141	.9512	30
40	225	831	715	177	393	231	124	483	20
50	254	854	708	221	385	233	107	454	10
36° 00′	.6283	.5878	1.701	.7265	1.376	1.236	.8090	.9425	54° 00′
10	312	901	695	310	368	239	073	396	50
20	341	925	688	355	360	241	056	367	40
30	.6370	.5948	1.681	.7400	1.351	1.244	.8039	.9338	30
40	400	972	675	445	343	247	021	308	20
50	429	995	668	490	335	249	004	279	10
37° 00′	.6458	.6018	1.662	.7536	1.327	1.252	.7986	.9250	53° 00′
10	487	041	655	581	319	255	969	221	50
20	516	065	649	627	311	258	951	192	40
30	.6545	.6088	1.643	.7673	1.303	1.260	.7934	.9163	30
40	574	111	636	720	295	263	916	134	20
50	603	134	630	766	288	266	898	105	10
38° 00′	.6632	.6157	1.624	.7813	1.280	1.269	.7880	.9076	52° 00′
10	661	180	618	860	272	272	862	047	50
20	690	202	612	907	265	275	844	.9018	40
30	.6720	.6225	1.606	.7954	1.257	1.278	.7826	.8988	30
40	749	248	601	.8002	250	281	808	959	20
50	778	271	595	050	242	284	790	930	10
39° 00′	.6807	.6293	1.589	.8098	1.235	1.287	.7771	.8901	51° 00′
10	836	316	583	146	228	290	753	872	50
20	865	338	578	195	220	293	735	843	40
30	.6894	.6361	1.572	.8243	1.213	1.296	.7716	.8814	30
40	923	383	567	292	206	299	698	785	20
50	952	406	561	342	199	302	679	756	10
40° 00′	.6981	.6428	1.556	.8391	1.192	1.305	.7660	.8727	50° 00′
10	.7010	450	550	441	185	309	642	698	50
20	039	472	545	491	178	312	623	668	40
30	.7069	.6494	1.540	.8541	1.171	1.315	.7604	.8639	30
40	098	517	535	591	164	318	585	610	20
50	127	539	529	642	157	322	566	581	10
41° 00′	.7156	.6561	1.524	.8693	1.150	1.325	.7547	.8552	49° 00′
		cos θ	sec θ	cot θ	tan θ	csc θ	sin θ	Radians	Degrees
								Angle θ	

TABLE I—continued

Angle θ									
Degrees	Radians	sin θ	csc θ	tan θ	cot θ	sec θ	cos θ		
41° 00′	.7156	.6561	1.524	.8693	1.150	1.325	.7547	.8552	49° 00′
10	185	583	519	744	144	328	528	523	50
20	214	604	514	796	137	332	509	494	40
30	.7243	.6626	1.509	.8847	1.130	1.335	.7490	.8465	30
40	272	648	504	899	124	339	470	436	20
50	301	670	499	952	117	342	451	407	10
42° 00′	.7330	.6691	1.494	.9004	1.111	1.346	.7431	.8378	48° 00′
10	359	713	490	057	104	349	412	348	50
20	389	734	485	110	098	353	392	319	40
30	.7418	.6756	1.480	.9163	1.091	1.356	.7373	.8290	30
40	447	777	476	217	085	360	353	261	20
50	476	799	471	271	079	364	333	232	10
43° 00′	.7505	.6820	1.466	.9325	1.072	1.367	.7314	.8203	47° 00′
10	534	841	462	380	066	371	294	174	50
20	563	862	457	435	060	375	274	145	40
30	.7592	.6884	1.453	.9490	1.054	1.379	.7254	.8116	30
40	621	905	448	545	048	382	234	087	20
50	650	926	444	601	042	386	214	058	10
44° 00′	.7679	.6947	1.440	.9657	1.036	1.390	.7193	.8029	46° 00′
10	709	967	435	713	030	394	173	.7999	50
20	738	988	431	770	024	398	153	970	40
30	.7767	.7009	1.427	.9827	1.018	1.402	.7133	.7941	30
40	796	030	423	884	012	406	112	912	20
50	825	050	418	942	006	410	092	883	10
45° 00′	.7854	.7071	1.414	1.000	1.000	1.414	.7071	.7854	45° 00′
		cos θ	sec θ	cot θ	tan θ	csc θ	sin θ	Radians	Degrees
								Angle θ	

TABLE II

Four-Place Values of Trigonometric Functions
Real Numbers u, or Angles θ, in Radians and Degrees

Real Number u or θ radians	θ degrees	$\sin u$ or $\sin \theta$	$\csc u$ or $\csc \theta$	$\tan u$ or $\tan \theta$	$\cot u$ or $\cot \theta$	$\sec u$ or $\sec \theta$	$\cos u$ or $\cos \theta$
0.00	0° 00′	0.0000	No value	0.0000	No value	1.000	1.000
.01	0° 34′	.0100	100.0	.0100	100.0	1.000	1.000
.02	1° 09′	.0200	50.00	.0200	49.99	1.000	0.9998
.03	1° 43′	.0300	33.34	.0300	33.32	1.000	0.9996
.04	2° 18′	.0400	25.01	.0400	24.99	1.001	0.9992
0.05	2° 52′	0.0500	20.01	0.0500	19.98	1.001	0.9988
.06	3° 26′	.0600	16.68	.0601	16.65	1.002	.9982
.07	4° 01′	.0699	14.30	.0701	14.26	1.002	.9976
.08	4° 35′	.0799	12.51	.0802	12.47	1.003	.9968
.09	5° 09′	.0899	11.13	.0902	11.08	1.004	.9960
0.10	5° 44′	0.0998	10.02	0.1003	9.967	1.005	0.9950
.11	6° 18′	.1098	9.109	.1104	9.054	1.006	.9940
.12	6° 53′	.1197	8.353	.1206	8.293	1.007	.9928
.13	7° 27′	.1296	7.714	.1307	7.649	1.009	.9916
.14	8° 01′	.1395	7.166	.1409	7.096	1.010	.9902
0.15	8° 36′	0.1494	6.692	0.1511	6.617	1.011	0.9888
.16	9° 10′	.1593	6.277	.1614	6.197	1.013	.9872
.17	9° 44′	.1692	5.911	.1717	5.826	1.015	.9856
.18	10° 19′	.1790	5.586	.1820	5.495	1.016	.9838
.19	10° 53′	.1889	5.295	.1923	5.200	1.018	.9820
0.20	11° 28′	0.1987	5.033	0.2027	4.933	1.020	0.9801
.21	12° 02′	.2085	4.797	.2131	4.692	1.022	.9780
.22	12° 36′	.2182	4.582	.2236	4.472	1.025	.9759
.23	13° 11′	.2280	4.386	.2341	4.271	1.027	.9737
.24	13° 45′	.2377	4.207	.2447	4.086	1.030	.9713
0.25	14° 19′	0.2474	4.042	0.2553	3.916	1.032	0.9689
.26	14° 54′	.2571	3.890	.2660	3.759	1.035	.9664
.27	15° 28′	.2667	3.749	.2768	3.613	1.038	.9638
.28	16° 03′	.2764	3.619	.2876	3.478	1.041	.9611
.29	16° 37′	.2860	3.497	.2984	3.351	1.044	.9582
0.30	17° 11′	0.2955	3.384	0.3093	3.233	1.047	0.9553
.31	17° 46′	.3051	3.278	.3203	3.122	1.050	.9523
.32	18° 20′	.3146	3.179	.3314	3.018	1.053	.9492
.33	18° 54′	.3240	3.086	.3425	2.920	1.057	.9460
.34	19° 29′	.3335	2.999	.3537	2.827	1.061	.9428
0.35	20° 03′	0.3429	2.916	0.3650	2.740	1.065	0.9394
Real Number u or θ radians	θ degrees	$\sin u$ or $\sin \theta$	$\csc u$ or $\csc \theta$	$\tan u$ or $\tan \theta$	$\cot u$ or $\cot \theta$	$\sec u$ or $\sec \theta$	$\cos u$ or $\cos \theta$

TABLE II—*continued*

Real Number u or θ radians	θ degrees	sin u or sin θ	csc u or csc θ	tan u or tan θ	cot u or cot θ	sec u or sec θ	cos u or cos θ
0.35	20° 03′	0.3429	2.916	0.3650	2.740	1.065	0.9394
.36	20° 38′	.3523	2.839	.3764	2.657	1.068	.9359
.37	21° 12′	.3616	2.765	.3879	2.578	1.073	.9323
.38	21° 46′	.3709	2.696	.3994	2.504	1.077	.9287
.39	22° 21′	.3802	2.630	.4111	2.433	1.081	.9249
0.40	22° 55′	0.3894	2.568	0.4228	2.365	1.086	0.9211
.41	23° 29′	.3986	2.509	.4346	2.301	1.090	.9171
.42	24° 04′	.4078	2.452	.4466	2.239	1.095	.9131
.43	24° 38′	.4169	2.399	.4586	2.180	1.100	.9090
.44	25° 13′	.4259	2.348	.4708	2.124	1.105	.9048
0.45	25° 47′	0.4350	2.299	0.4831	2.070	1.111	0.9004
.46	26° 21′	.4439	2.253	.4954	2.018	1.116	.8961
.47	26° 56′	.4529	2.208	.5080	1.969	1.122	.8916
.48	27° 30′	.4618	2.166	.5206	1.921	1.127	.8870
.49	28° 04′	.4706	2.125	.5334	1.875	1.133	.8823
0.50	28° 39′	0.4794	2.086	0.5463	1.830	1.139	0.8776
.51	29° 13′	.4882	2.048	.5594	1.788	1.146	.8727
.52	29° 48′	.4969	2.013	.5726	1.747	1.152	.8678
.53	30° 22′	.5055	1.978	.5859	1.707	1.159	.8628
.54	30° 56′	.5141	1.945	.5994	1.668	1.166	.8577
0.55	31° 31′	0.5227	1.913	0.6131	1.631	1.173	0.8525
.56	32° 05′	.5312	1.883	.6269	1.595	1.180	.8473
.57	32° 40′	.5396	1.853	.6410	1.560	1.188	.8419
.58	33° 14′	.5480	1.825	.6552	1.526	1.196	.8365
.59	33° 48′	.5564	1.797	.6696	1.494	1.203	.8309
0.60	34° 23′	0.5646	1.771	0.6841	1.462	1.212	0.8253
.61	34° 57′	.5729	1.746	.6989	1.431	1.220	.8196
.62	35° 31′	.5810	1.721	.7139	1.401	1.229	.8139
.63	36° 06′	.5891	1.697	.7291	1.372	1.238	.8080
.64	36° 40′	.5972	1.674	.7445	1.343	1.247	.8021
0.65	37° 15′	0.6052	1.652	0.7602	1.315	1.256	0.7961
.66	37° 49′	.6131	1.631	.7761	1.288	1.266	.7900
.67	38° 23′	.6210	1.610	.7923	1.262	1.276	.7838
.68	38° 58′	.6288	1.590	.8087	1.237	1.286	.7776
.69	39° 32′	.6365	1.571	.8253	1.212	1.297	.7712
0.70	40° 06′	0.6442	1.552	0.8423	1.187	1.307	0.7648
Real Number u or θ radians	θ degrees	sin u or sin θ	csc u or csc θ	tan u or tan θ	cot u or cot θ	sec u or sec θ	cos u or cos θ

TABLE II—*continued*

Real Number u or θ radians	θ degrees	sin u or sin θ	csc u or csc θ	tan u or tan θ	cot u or cot θ	sec u or sec θ	cos u or cos θ
0.70	40° 06′	0.6442	1.552	0.8423	1.187	1.307	0.7648
.71	40° 41′	.6518	1.534	.8595	1.163	1.319	.7584
.72	41° 15′	.6594	1.517	.8771	1.140	1.330	.7518
.73	41° 50′	.6669	1.500	.8949	1.117	1.342	.7452
.74	42° 24′	.6743	1.483	.9131	1.095	1.354	.7385
0.75	42° 58′	0.6816	1.467	0.9316	1.073	1.367	0.7317
.76	43° 33′	.6889	1.452	.9505	1.052	1.380	.7248
.77	44° 07′	.6961	1.436	.9697	1.031	1.393	.7179
.78	44° 41′	.7033	1.422	.9893	1.011	1.407	.7109
.79	45° 16′	.7104	1.408	1.009	.9908	1.421	.7038
0.80	45° 50′	0.7174	1.394	1.030	0.9712	1.435	0.6967
.81	46° 25′	.7243	1.381	1.050	.9520	1.450	.6895
.82	46° 59′	.7311	1.368	1.072	.9331	1.466	.6822
.83	47° 33′	.7379	1.355	1.093	.9146	1.482	.6749
.84	48° 08′	.7446	1.343	1.116	.8964	1.498	.6675
0.85	48° 42′	0.7513	1.331	1.138	0.8785	1.515	0.6600
.86	49° 16′	.7578	1.320	1.162	.8609	1.533	.6524
.87	49° 51′	.7643	1.308	1.185	.8437	1.551	.6448
.88	50° 25′	.7707	1.297	1.210	.8267	1.569	.6372
.89	51° 00′	.7771	1.287	1.235	.8100	1.589	.6294
0.90	51° 34′	0.7833	1.277	1.260	0.7936	1.609	0.6216
.91	52° 08′	.7895	1.267	1.286	.7774	1.629	.6137
.92	52° 43′	.7956	1.257	1.313	.7615	1.651	.6058
.93	53° 17′	.8016	1.247	1.341	.7458	1.673	.5978
.94	53° 51′	.8076	1.238	1.369	.7303	1.696	.5898
0.95	54° 26′	0.8134	1.229	1.398	0.7151	1.719	0.5817
.96	55° 00′	.8192	1.221	1.428	.7001	1.744	.5735
.97	55° 35′	.8249	1.212	1.459	.6853	1.769	.5653
.98	56° 09′	.8305	1.204	1.491	.6707	1.795	.5570
.99	56° 43′	.8360	1.196	1.524	.6563	1.823	.5487
1.00	57° 18′	0.8415	1.188	1.557	0.6421	1.851	0.5403
1.01	57° 52′	.8468	1.181	1.592	.6281	1.880	.5319
1.02	58° 27′	.8521	1.174	1.628	.6142	1.911	.5234
1.03	59° 01′	.8573	1.166	1.665	.6005	1.942	.5148
1.04	59° 35′	.8624	1.160	1.704	.5870	1.975	.5062
1.05	60° 10′	0.8674	1.153	1.743	0.5736	2.010	0.4976
Real Number u or θ radians	θ degrees	sin u or sin θ	csc u or csc θ	tan u or tan θ	cot u or cot θ	sec u or sec θ	cos u or cos θ

TABLE II—*continued*

Real Number u or θ radians	θ degrees	sin u or sin θ	csc u or csc θ	tan u or tan θ	cot u or cot θ	sec u or sec θ	cos u or cos θ
1.05	60° 10′	0.8674	1.153	1.743	0.5736	2.010	0.4976
1.06	60° 44′	.8724	1.146	1.784	.5604	2.046	.4889
1.07	61° 18′	.8772	1.140	1.827	.5473	2.083	.4801
1.08	61° 53′	.8820	1.134	1.871	.5344	2.122	.4713
1.09	62° 27′	.8866	1.128	1.917	.5216	2.162	.4625
1.10	63° 02′	0.8912	1.122	1.965	0.5090	2.205	0.4536
1.11	63° 36′	.8957	1.116	2.014	.4964	2.249	.4447
1.12	64° 10′	.9001	1.111	2.066	.4840	2.295	.4357
1.13	64° 45′	.9044	1.106	2.120	.4718	2.344	.4267
1.14	65° 19′	.9086	1.101	2.176	.4596	2.395	.4176
1.15	65° 53′	0.9128	1.096	2.234	0.4475	2.448	0.4085
1.16	66° 28′	.9168	1.091	2.296	.4356	2.504	.3993
1.17	67° 02′	.9208	1.086	2.360	.4237	2.563	.3902
1.18	67° 37′	.9246	1.082	2.247	.4120	2.625	.3809
1.19	68° 11′	.9284	1.077	2.498	.4003	2.691	.3717
1.20	68° 45′	0.9320	1.073	2.572	0.3888	2.760	0.3624
1.21	69° 20′	.9356	1.069	2.650	.3773	2.833	.3530
1.22	69° 54′	.9391	1.065	2.733	.3659	2.910	.3436
1.23	70° 28′	.9425	1.061	2.820	.3546	2.992	.3342
1.24	71° 03′	.9458	1.057	2.912	.3434	3.079	.3248
1.25	71° 37′	0.9490	1.054	3.010	0.3323	3.171	0.3153
1.26	72° 12′	.9521	1.050	3.113	.3212	3.270	.3058
1.27	72° 46′	.9551	1.047	3.224	.3102	3.375	.2963
1.28	73° 20′	.9580	1.044	3.341	.2993	3.488	.2867
1.29	73° 55′	.9608	1.041	3.467	.2884	3.609	.2771
1.30	74° 29′	0.9636	1.038	3.602	0.2776	3.738	0.2675
1.31	75° 03′	.9662	1.035	3.747	.2669	3.878	.2579
1.32	75° 38′	.9687	1.032	3.903	.2562	4.029	.2482
1.33	76° 12′	.9711	1.030	4.072	.2456	4.193	.2385
1.34	76° 47′	.9735	1.027	4.256	.2350	4.372	.2288
1.35	77° 21′	0.9757	1.025	4.455	0.2245	4.566	0.2190
1.36	77° 55′	.9779	1.023	4.673	.2140	4.779	.2092
1.37	78° 30′	.9799	1.021	4.913	.2035	5.014	.1994
1.38	79° 04′	.9819	1.018	5.177	.1931	5.273	.1896
1.39	79° 38′	.9837	1.017	5.471	.1828	5.561	.1798
1.40	80° 13′	0.9854	1.015	5.798	0.1725	5.883	0.1700
Real Number u or θ radians	θ degrees	sin u or sin θ	csc u or csc θ	tan u or tan θ	cot u or cot θ	sec u or sec θ	cos u or cos θ

TABLE II—*continued*

Real Number u or θ radians	θ degrees	sin u or sin θ	csc u or csc θ	tan u or tan θ	cot u or cot θ	sec u or sec θ	cos u or cos θ
1.40	80° 13′	0.9854	1.015	5.798	0.1725	5.883	0.1700
1.41	80° 47′	.9871	1.013	6.165	.1622	6.246	.1601
1.42	81° 22′	.9887	1.011	6.581	.1519	6.657	.1502
1.43	81° 56′	.9901	1.010	7.055	.1417	7.126	.1403
1.44	82° 30′	.9915	1.009	7.602	.1315	7.667	.1304
1.45	83° 05′	0.9927	1.007	8.238	0.1214	8.299	0.1205
1.46	83° 39′	.9939	1.006	8.989	.1113	9.044	.1106
1.47	84° 13′	.9949	1.005	9.887	.1011	9.938	.1006
1.48	84° 48′	.9959	1.004	10.98	.0910	11.03	.0907
1.49	85° 22′	.9967	1.003	12.35	.0810	12.39	.0807
1.50	85° 57′	0.9975	1.003	14.10	0.0709	14.14	0.0707
1.51	86° 31′	.9982	1.002	16.43	.0609	16.46	.0608
1.52	87° 05′	.9987	1.001	19.67	.0508	19.69	.0508
1.53	87° 40′	.9992	1.001	24.50	.0408	24.52	.0408
1.54	88° 14′	.9995	1.000	32.46	.0308	32.48	.0308
1.55	88° 49′	0.9998	1.000	48.08	0.0208	48.09	0.0208
1.56	89° 23′	.9999	1.000	92.62	.0108	92.63	.0108
1.57	89° 57′	1.000	1.000	1256	.0008	1256	.0008
Real Number u or θ radians	θ degrees	sin u or sin θ	csc u or csc θ	tan u or tan θ	cot u or cot θ	sec u or sec θ	cos u or cos θ

TABLE III

Four-Place Logarithms of Numbers from 1 to 10
To extend the table write the number N as
$N = n \times 10^c$, $1 \leq n < 10$, c an integer, and use
$\log N = \log n + c$.

n	0	1	2	3	4	5	6	7	8	9
1.0	+0.0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
1.1	.0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
1.2	.0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
1.3	.1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
1.4	.1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
1.5	.1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
1.6	.2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
1.7	.2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
1.8	.2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
1.9	.2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
2.0	.3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
2.1	.3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
2.2	.3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
2.3	.3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
2.4	.3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
2.5	.3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
2.6	.4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
2.7	.4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
2.8	.4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
2.9	.4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
3.0	.4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
3.1	.4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
3.2	.5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
3.3	.5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
3.4	.5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
3.5	.5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
3.6	.5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
3.7	.5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
3.8	.5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
3.9	.5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
4.0	.6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
4.1	.6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
4.2	.6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
4.3	.6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
4.4	.6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
4.5	.6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
4.6	.6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
4.7	.6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
4.8	.6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
4.9	.6902	6911	6920	6928	6937	6946	6955	6964	6972	6981

TABLE III—*continued*

n	0	1	2	3	4	5	6	7	8	9
5.0	+.6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
5.1	.7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
5.2	.7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
5.3	.7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
5.4	.7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
5.5	.7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
5.6	.7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
5.7	.7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
5.8	.7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
5.9	.7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
6.0	.7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
6.1	.7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
6.2	.7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
6.3	.7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
6.4	.8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
6.5	.8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
6.6	.8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
6.7	.8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
6.8	.8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
6.9	.8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
7.0	.8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
7.1	.8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
7.2	.8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
7.3	.8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
7.4	.8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
7.5	.8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
7.6	.8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
7.7	.8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
7.8	.8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
7.9	.8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
8.0	.9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
8.1	.9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
8.2	.9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
8.3	.9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
8.4	.9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
8.5	.9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
8.6	.9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
8.7	.9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
8.8	.9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
8.9	.9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
9.0	.9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
9.1	.9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
9.2	.9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
9.3	.9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
9.4	.9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
9.5	.9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
9.6	.9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
9.7	.9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
9.8	.9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
9.9	.9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

TABLE IV

Four-Place Logarithms of Trigonometric Functions
Angle θ in Degrees

Attach − 10 to Logarithms Obtained from This Table

Angle θ	L sin θ	L csc θ	L tan θ	L cot θ	L sec θ	L cos θ	
0° 00′	No value	No value	No value	No value	10.0000	10.0000	90° 00′
10′	7.4637	12.5363	7.4637	12.5363	.0000	.0000	50′
20′	.7648	.2352	.7648	.2352	.0000	.0000	40′
30′	7.9408	12.0592	7.9409	12.0591	.0000	.0000	30′
40′	8.0658	11.9342	8.0658	11.9342	.0000	.0000	20′
50′	.1627	.8373	.1627	.8373	.0000	10.0000	10′
1° 00′	8.2419	11.7581	8.2419	11.7581	10.0001	9.9999	89° 00′
10′	.3088	.6912	.3089	.6911	.0001	.9999	50′
20′	.3668	.6332	.3669	.6331	.0001	.9999	40′
30′	.4179	.5821	.4181	.5819	.0001	.9999	30′
40′	.4637	.5363	.4638	.5362	.0002	.9998	20′
50′	.5050	.4950	.5053	.4947	.0002	.9998	10′
2° 00′	8.5428	11.4572	8.5431	11.4569	10.0003	9.9997	88° 00′
10′	.5776	.4224	.5779	.4221	.0003	.9997	50′
20′	.6097	.3903	.6101	.3899	.0004	.9996	40′
30′	.6397	.3603	.6401	.3599	.0004	.9996	30′
40′	.6677	.3323	.6682	.3318	.0005	.9995	20′
50′	.6940	.3060	.6945	.3055	.0005	.9995	10′
3° 00′	8.7188	11.2812	8.7194	11.2806	10.0006	9.9994	87° 00′
10′	.7423	.2577	.7429	.2571	.0007	.9993	50′
20′	.7645	.2355	.7652	.2348	.0007	.9993	40′
30′	.7857	.2143	.7865	.2135	.0008	.9992	30′
40′	.8059	.1941	.8067	.1933	.0009	.9991	20′
50′	.8251	.1749	.8261	.1739	.0010	.9990	10′
4° 00′	8.8436	11.1564	8.8446	11.1554	10.0011	9.9989	86° 00′
10′	.8613	.1387	.8624	.1376	.0011	.9989	50′
20′	.8783	.1217	.8795	.1205	.0012	.9988	40′
30′	.8946	.1054	.8960	.1040	.0013	.9987	30′
40′	.9104	.0896	.9118	.0882	.0014	.9986	20′
50′	.9256	.0744	.9272	.0728	.0015	.9985	10′
5° 00′	8.9403	11.0597	8.9420	11.0580	10.0017	9.9983	85° 00′
10′	.9545	.0455	.9563	.0437	.0018	.9982	50′
20′	.9682	.0318	.9701	.0299	.0019	.9981	40′
30′	.9816	.0184	.9836	.0164	.0020	.9980	30′
40′	8.9945	11.0055	8.9966	11.0034	.0021	.9979	20′
50′	9.0070	10.9930	9.0093	10.9907	.0023	.9977	10′
6° 00′	9.0192	10.9808	9.0216	10.9784	10.0024	9.9976	84° 00′
	L cos θ	L sec θ	L cot θ	L tan θ	L csc θ	L sin θ	Angle θ

TABLE IV—continued

Attach −10 to Logarithms Obtained from This Table

Angle θ	L sin θ	L csc θ	L tan θ	L cot θ	L sec θ	L cos θ	
6° 00′	9.0192	10.9808	9.0216	10.9784	10.0024	9.9976	84° 00′
10′	.0311	.9689	.0336	.9664	.0025	.9975	50′
20′	.0426	.9574	.0453	.9547	.0027	.9973	40′
30′	.0539	.9461	.0567	.9433	.0028	.9972	30′
40′	.0648	.9352	.0678	.9322	.0029	.9971	20′
50′	.0755	.9245	.0786	.9214	.0031	.9969	10′
7° 00′	9.0859	10.9141	9.0891	10.9109	10.0032	9.9968	83° 00′
10′	.0961	.9039	.0995	.9005	.0034	.9966	50′
20′	.1060	.8940	.1096	.8904	.0036	.9964	40′
30′	.1157	.8843	.1194	.8806	.0037	.9963	30′
40′	.1252	.8748	.1291	.8709	.0039	.9961	20′
50′	.1345	.8655	.1385	.8615	.0041	.9959	10′
8° 00′	9.1436	10.8564	9.1478	10.8522	10.0042	9.9958	82° 00′
10′	.1525	.8475	.1569	.8431	.0044	.9956	50′
20′	.1612	.8388	.1658	.8342	.0046	.9954	40′
30′	.1697	.8303	.1745	.8255	.0048	.9952	30′
40′	.1781	.8219	.1831	.8169	.0050	.9950	20′
50′	.1863	.8137	.1915	.8085	.0052	.9948	10′
9° 00′	9.1943	10.8057	9.1997	10.8003	10.0054	9.9946	81° 00′
10′	.2022	.7978	.2078	.7922	.0056	.9944	50′
20′	.2100	.7900	.2158	.7842	.0058	.9942	40′
30′	.2176	.7824	.2236	.7764	.0060	.9940	30′
40′	.2251	.7749	.2313	.7687	.0062	.9938	20′
50′	.2324	.7676	.2389	.7611	.0064	.9936	10′
10° 00′	9.2397	10.7603	9.2463	10.7537	10.0066	9.9934	80° 00′
10′	.2468	.7532	.2536	.7464	.0069	.9931	50′
20′	.2538	.7462	.2609	.7391	.0071	.9929	40′
30′	.2606	.7394	.2680	.7320	.0073	.9927	30′
40′	.2674	.7326	.2750	.7250	.0076	.9924	20′
50′	.2740	.7260	.2819	.7181	.0078	.9922	10′
11° 00′	9.2806	10.7194	9.2887	10.7113	10.0081	9.9919	79° 00′
10′	.2870	.7130	.2953	.7047	.0083	.9917	50′
20′	.2934	.7066	.3020	.6980	.0086	.9914	40′
30′	.2997	.7003	.3085	.6915	.0088	.9912	30′
40′	.3058	.6942	.3149	.6851	.0091	.9909	20′
50′	.3119	.6881	.3212	.6788	.0093	.9907	10′
12° 00′	9.3179	10.6821	9.3275	10.6725	10.0096	9.9904	78° 00′
10′	.3238	.6762	.3336	.6664	.0099	.9901	50′
20′	.3296	.6704	.3397	.6603	.0101	.9899	40′
30′	.3353	.6647	.3458	.6542	.0104	.9896	30′
40′	.3410	.6590	.3517	.6483	.0107	.9893	20′
50′	.3466	.6534	.3576	.6424	.0110	.9890	10′
13° 00′	9.3521	10.6479	9.3634	10.6366	10.0113	9.9887	77° 00′
	L cos θ	L sec θ	L cot θ	L tan θ	L csc θ	L sin θ	Angle θ

TABLE IV—*continued*

Attach − 10 to Logarithms Obtained from This Table

Angle θ	L sin θ	L csc θ	L tan θ	L cot θ	L sec θ	L cos θ	
13° 00′	9.3521	10.6479	9.3634	10.6366	10.0113	9.9887	77° 00′
10′	.3575	.6425	.3691	.6309	.0116	.9884	50′
20′	.3629	.6371	.3748	.6252	.0119	.9881	40′
30′	.3682	.6318	.3804	.6196	.0122	.9878	30′
40′	.3734	.6266	.3859	.6141	.0125	.9875	20′
50′	.3786	.6214	.3914	.6086	.0128	.9872	10′
14° 00′	9.3837	10.6163	9.3968	10.6032	10.0131	9.9869	76° 00′
10′	.3887	.6113	.4021	.5979	.0134	.9866	50′
20′	.3937	.6063	.4074	.5926	.0137	.9863	40′
30′	.3986	.6014	.4127	.5873	.0141	.9859	30′
40′	.4035	.5965	.4178	.5822	.0144	.9856	20′
50′	.4083	.5917	.4230	.5770	.0147	.9853	10′
15° 00′	9.4130	10.5870	9.4281	10.5719	10.0151	9.9849	75° 00′
10′	.4177	.5823	.4331	.5669	.0154	.9846	50′
20′	.4223	.5777	.4381	.5619	.0157	.9843	40′
30′	.4269	.5731	.4430	.5570	.0161	.9839	30′
40′	.4314	.5686	.4479	.5521	.0164	.9836	20′
50′	.4359	.5641	.4527	.5473	.0168	.9832	10′
16° 00′	9.4403	10.5597	9.4575	10.5425	10.0172	9.9828	74° 00′
10′	.4447	.5553	.4622	.5378	.0175	.9825	50′
20′	.4491	.5509	.4669	.5331	.0179	.9821	40′
30′	.4533	.5467	.4716	.5284	.0183	.9817	30′
40′	.4576	.5424	.4762	.5238	.0186	.9814	20′
50′	.4618	.5382	.4808	.5192	.0190	.9810	10′
17° 00′	9.4659	10.5341	9.4853	10.5147	10.0194	9.9806	73° 00′
10′	.4700	.5300	.4898	.5102	.0198	.9802	50′
20′	.4741	.5259	.4943	.5057	.0202	.9798	40′
30′	.4781	.5219	.4987	.5013	.0206	.9794	30′
40′	.4821	.5179	.5031	.4969	.0210	.9790	20′
50′	.4861	.5139	.5075	.4925	.0214	.9786	10′
18° 00′	9.4900	10.5100	9.5118	10.4882	10.0218	9.9782	72° 00′
10′	.4939	.5061	.5161	.4839	.0222	.9778	50′
20′	.4977	.5023	.5203	.4797	.0226	.9774	40′
30′	.5015	.4985	.5245	.4755	.0230	.9770	30′
40′	.5052	.4948	.5287	.4713	.0235	.9765	20′
50′	.5090	.4910	.5329	.4671	.0239	.9761	10′
19° 00′	9.5126	10.4874	9.5370	10.4630	10.0243	9.9757	71° 00′
10′	.5163	.4837	.5411	.4589	.0248	.9752	50′
20′	.5199	.4801	.5451	.4549	.0252	.9748	40′
30′	.5235	.4765	.5491	.4509	.0257	.9743	30′
40′	.5270	.4730	.5531	.4469	.0261	.9739	20′
50′	.5306	.4694	.5571	.4429	.0266	.9734	10′
20° 00′	9.5341	10.4659	9.5611	10.4389	10.0270	9.9730	70° 00′
	L cos θ	L sec θ	L cot θ	L tan θ	L csc θ	L sin θ	Angle θ

TABLE IV—*continued*

Attach – 10 to Logarithms Obtained from This Table

Angle θ	L sin θ	L csc θ	L tan θ	L cot θ	L sec θ	L cos θ	
20° 00′	9.5341	10.4659	9.5611	10.4389	10.0270	9.9730	70° 00′
10′	.5375	.4625	.5650	.4350	.0275	.9725	50′
20′	.5409	.4591	.5689	.4311	.0279	.9721	40′
30′	.5443	.4557	.5727	.4273	.0284	.9716	30′
40′	.5477	.4523	.5766	.4234	.0289	.9711	20′
50′	.5510	.4490	.5804	.4196	.0294	.9706	10′
21° 00′	9.5543	10.4457	9.5842	10.4158	10.0298	9.9702	69° 00′
10′	.5576	.4424	.5879	.4121	.0303	.9797	50′
20′	.5609	.4391	.5917	.4083	.0308	.9692	40′
30′	.5641	.4359	.5954	.4046	.0313	.9687	30′
40′	.5673	.4327	.5991	.4009	.0318	.9682	20′
50′	.5704	.4296	.6028	.3972	.0323	.9677	10′
22° 00′	9.5736	10.4264	9.6064	10.3936	10.0328	9.9672	68° 00′
10′	.5767	.4233	.6100	.3900	.0333	.9667	50′
20′	.5798	.4202	.6136	.3864	.0339	.9661	40′
30′	.5828	.4172	.6172	.3828	.0344	.9656	30′
40′	.5859	.4141	.6208	.3792	.0349	.9651	20′
50′	.5889	.4111	.6243	.3757	.0354	.9646	10′
23° 00′	9.5919	10.4081	9.6279	10.3721	10.0360	9.9640	67° 00′
10′	.5948	.4052	.6314	.3686	.0365	.9635	50′
20′	.5978	.4022	.6348	.3652	.0371	.9629	40′
30′	.6007	.3993	.6383	.3617	.0376	.9624	30′
40′	.6036	.3964	.6417	.3583	.0382	.9618	20′
50′	.6065	.3935	.6452	.3548	.0387	.9613	10′
24° 00′	9.6093	10.3907	9.6486	10.3514	10.0393	9.9607	66° 00′
10′	.6121	.3879	.6520	.3480	.0398	.9602	50′
20′	.6149	.3851	.6553	.3447	.0404	.9596	40′
30′	.6177	.3823	.6587	.3413	.0410	.9590	30′
40′	.6205	.3795	.6620	.3380	.0416	.9584	20′
50′	.6232	.3768	.6654	.3346	.0421	.9579	10′
25° 00′	9.6259	10.3741	9.6687	10.3313	10.0427	9.9573	65° 00′
10′	.6286	.3714	.6720	.3280	.0433	.9567	50′
20′	.6313	.3687	.6752	.3248	.0439	.9561	40′
30′	.6340	.3660	.6785	.3215	.0445	.9555	30′
40′	.6366	.3634	.6817	.3183	.0451	.9549	20′
50′	.6392	.3608	.6850	.3150	.0457	.9543	10′
26° 00′	9.6418	10.3582	9.6882	10.3118	10.0463	9.9537	64° 00′
10′	.6444	.3556	.6914	.3086	.0470	.9530	50′
20′	.6470	.3530	.6946	.3054	.0476	.9524	40′
30′	.6495	.3505	.6977	.3023	.0482	.9518	30′
40′	.6521	.3479	.7009	.2991	.0488	.9512	20′
50′	.6546	.3454	.7040	.2960	.0495	.9505	10′
27° 00′	9.6570	10.3430	9.7072	10.2928	10.0501	9.9499	63° 00′
	L cos θ	L sec θ	L cot θ	L tan θ	L csc θ	L sin θ	Angle θ

TABLE IV—*continued*

Attach − 10 to Logarithms Obtained from This Table

Angle θ	L sin θ	L csc θ	L tan θ	L cot θ	L sec θ	L cos θ	
27° 00′	9.6570	10.3430	9.7072	10.2928	10.0501	9.9499	63° 00′
10′	.6595	.3405	.7103	.2897	.0508	.9492	50′
20′	.6620	.3380	.7134	.2866	.0514	.9486	40′
30′	.6644	.3356	.7165	.2835	.0521	.9479	30′
40′	.6668	.3332	.7196	.2804	.0527	.9473	20′
50′	.6692	.3308	.7226	.2774	.0534	.9466	10′
28° 00′	9.6716	10.3284	9.7257	10.2743	10.0541	9.9459	62° 00′
10′	.6740	.3260	.7287	.2713	.0547	.9453	50′
20′	.6763	.3237	.7317	.2683	.0554	.9446	40′
30′	.6787	.3213	.7348	.2652	.0561	.9439	30′
40′	.6810	.3190	.7378	.2622	.0568	.9432	20′
50′	.6833	.3167	.7408	.2592	.0575	.9425	10′
29° 00′	9.6856	10.3144	9.7438	10.2562	10.0582	9.9418	61° 00′
10′	.6878	.3122	.7467	.2533	.0589	.9411	50′
20′	.6901	.3099	.7497	.2503	.0596	.9404	40′
30′	.6923	.3077	.7526	.2474	.0603	.9397	30′
40′	.6946	.3054	.7556	.2444	.0610	.9390	20′
50′	.6968	.3032	.7585	.2415	.0617	.9383	10′
30° 00′	9.6990	10.3010	9.7614	10.2386	10.0625	9.9375	60° 00′
10′	.7012	.2988	.7644	.2356	.0632	.9368	50′
20′	.7033	.2967	.7673	.2327	.0639	.9361	40′
30′	.7055	.2945	.7701	.2299	.0647	.9353	30′
40′	.7076	.2924	.7730	.2270	.0654	.9346	20′
50′	.7097	.2903	.7759	.2241	.0662	.9338	10′
31° 00′	9.7118	10.2882	9.7788	10.2212	10.0669	9.9331	59° 00′
10′	.7139	.2861	.7816	.2184	.0677	.9323	50′
20′	.7160	.2840	.7845	.2155	.0685	.9315	40′
30′	.7181	.2819	.7873	.2127	.0692	.9308	30′
40′	.7201	.2799	.7902	.2098	.0700	.9300	20′
50′	.7222	.2778	.7930	.2070	.0708	.9292	10′
32° 00′	9.7242	10.2758	9.7958	10.2042	10.0716	9.9284	58° 00′
10′	.7262	.2738	.7986	.2014	.0724	.9276	50′
20′	.7282	.2718	.8014	.1986	.0732	.9268	40′
30′	.7302	.2698	.8042	.1958	.0740	.9260	30′
40′	.7322	.2678	.8070	.1930	.0748	.9252	20′
50′	.7342	.2658	.8097	.1903	.0756	.9244	10′
33° 00′	9.7361	10.2639	9.8125	10.1875	10.0764	9.9236	57° 00′
10′	.7380	.2620	.8153	.1847	.0772	.9228	50′
20′	.7400	.2600	.8180	.1820	.0781	.9219	40′
30′	.7419	.2581	.8208	.1792	.0789	.9211	30′
40′	.7438	.2562	.8235	.1765	.0797	.9203	20′
50′	.7457	.2543	.8263	.1737	.0806	.9194	10′
34° 00′	9.7476	10.2524	9.8290	10.1710	10.0814	9.9186	56° 00′
	L cos θ	L sec θ	L cot θ	L tan θ	L csc θ	L sin θ	Angle θ

TABLE IV—*continued*

Attach − 10 to Logarithms Obtained from This Table

Angle θ	L sin θ	L csc θ	L tan θ	L cot θ	L sec θ	L cos θ	
34° 00′	9.7476	10.2524	9.8290	10.1710	10.0814	9.9186	**56° 00′**
10′	.7494	.2506	.8317	.1683	.0823	.9177	50′
20′	.7513	.2487	.8344	.1656	.0831	.9169	40′
30′	.7531	.2469	.8371	.1629	.0840	.9160	30′
40′	.7550	.2450	.8398	.1602	.0849	.9151	20′
50′	.7568	.2432	.8425	.1575	.0858	.9142	10′
35° 00′	9.7586	10.2414	9.8452	10.1548	10.0866	9.9134	**55° 00′**
10′	.7604	.2396	.8479	.1521	.0875	.9125	50′
20′	.7622	.2378	.8506	.1494	.0884	.9116	40′
30′	.7640	.2360	.8533	.1467	.0893	.9107	30′
40′	.7657	.2343	.8559	.1441	.0902	.9098	20′
50′	.7675	.2325	.8586	.1414	.0911	.9089	10′
36° 00′	9.7692	10.2308	9.8613	10.1387	10.0920	9.9080	**54° 00′**
10′	.7710	.2290	.8639	.1361	.0930	.9070	50′
20′	.7727	.2273	.8666	.1334	.0939	.9061	40′
30′	.7744	.2256	.8692	.1308	.0948	.9052	30′
40′	.7761	.2239	.8718	.1282	.0958	.9042	20′
50′	.7778	.2222	.8745	.1255	.0967	.9033	10′
37° 00′	9.7795	10.2205	9.8771	10.1229	10.0977	9.9023	**53° 00′**
10′	.7811	.2189	.8797	.1203	.0986	.9014	50′
20′	.7828	.2172	.8824	.1176	.0996	.9004	40′
30′	.7844	.2156	.8850	.1150	.1005	.8995	30′
40′	.7861	.2139	.8876	.1124	.1015	.8985	20′
50′	.7877	.2123	.8902	.1098	.1025	.8975	10′
38° 00′	9.7893	10.2107	9.8928	10.1072	10.1035	9.8965	**52° 00′**
10′	.7910	.2090	.8954	.1046	.1045	.8955	50′
20′	.7926	.2074	.8980	.1020	.1055	.8945	40′
30′	.7941	.2059	.9006	.0994	.1065	.8935	30′
40′	.7957	.2043	.9032	.0968	.1075	.8925	20′
50′	.7973	.2027	.9058	.0942	.1085	.8915	10′
39° 00′	9.7989	10.2011	9.9084	10.0916	10.1095	9.8905	**51° 00′**
10′	.8004	.1996	.9110	.0890	.1105	.8895	50′
20′	.8020	.1980	.9135	.0865	.1116	.8884	40′
30′	.8035	.1965	.9161	.0839	.1126	.8874	30′
40′	.8050	.1950	.9187	.0813	.1136	.8864	20′
50′	.8066	.1934	.9212	.0788	.1147	.8853	10′
40° 00′	9.8081	10.1919	9.9238	10.0762	10.1157	9.8843	**50° 00′**
10′	.8096	.1904	.9264	.0736	.1168	.8832	50′
20′	.8111	.1889	.9289	.0711	.1179	.8821	40′
30′	.8125	.1875	.9315	.0685	.1190	.8810	30′
40′	.8140	.1860	.9341	.0659	.1200	.8800	20′
50′	.8155	.1845	.9366	.0634	.1211	.8789	10′
41° 00′	9.8169	10.1831	9.9392	10.0608	10.1222	9.8778	**49° 00′**
	L cos θ	L sec θ	L cot θ	L tan θ	L csc θ	L sin θ	Angle θ

TABLE IV—*continued*

Attach −10 to Logarithms Obtained from This Table

Angle θ	L sin θ	L csc θ	L tan θ	L cot θ	L sec θ	L cos θ	
41° 00′	9.8169	10.1831	9.9392	10.0608	10.1222	9.8778	49° 00′
10′	.8184	.1816	.9417	.0583	.1233	.8767	50′
20′	.8198	.1802	.9443	.0557	.1244	.8756	40′
30′	.8213	.1787	.9468	.0532	.1255	.8745	30′
40′	.8227	.1773	.9494	.0506	.1267	.8733	20′
50′	.8241	.1759	.9519	.0481	.1278	.8722	10′
42° 00′	9.8255	10.1745	9.9544	10.0456	10.1289	9.8711	48° 00′
10′	.8269	.1731	.9570	.0430	.1301	.8699	50′
20′	.8283	.1717	.9595	.0405	.1312	.8688	40
30′	.8297	.1703	.9621	.0379	.1324	.8676	30′
40′	.8311	.1689	.9646	.0354	.1335	.8665	20′
50′	.8324	.1676	.9671	.0329	.1347	.8653	10′
43° 00′	9.8338	10.1662	9.9697	10.0303	10.1359	9.8641	47° 00′
10′	.8351	.1649	.9722	.0278	.1371	.8629	50′
20′	.8365	.1635	.9747	.0253	.1382	.8618	40′
30′	.8378	.1622	.9772	.0228	.1394	.8606	30′
40′	.8391	.1609	.9798	.0202	.1406	.8594	20′
50′	.8405	.1595	.9823	.0177	.1418	.8582	10′
44° 00′	9.8418	10.1582	9.9848	10.0152	10.1431	9.8569	46° 00′
10′	.8431	.1569	.9874	.0126	.1443	.8557	50′
20′	.8444	.1556	.9899	.0101	.1455	.8545	40′
30′	.8457	.1543	.9924	.0076	.1468	.8532	30′
40′	.8469	.1531	.9949	.0051	.1480	.8520	20′
50′	.8482	.1518	9.9975	.0025	.1493	.8507	10′
45° 00′	9.8495	10.1505	10.0000	10.0000	10.1505	9.8495	45° 00′
	L cos θ	L sec θ	L cot θ	L tan θ	L csc θ	L sin θ	Angle θ

ANSWERS

EXERCISE 1.1

3. $\{(1,4),(1,5),(2,4),(2,5),(3,4),(3,5),(4,5)\}$.
Inverse relation corresponds to the cartesian product
$\{(4,1),(5,1),(4,2),(5,2),(4,3),(5,3),(5,4)\}$ and corresponds to 'greater than' from B to A.

5. (ii)

7. $f^{-1} = \{(b,a),(d,b),(a,c),(c,d)\}$

8. Commutative and associative

9. Yes.

11. (iii) 2^n

12. (i) $n \to n^2 : N \to N$ is one-to-one but *not* onto.
(ii) $n \to |n| : Z \to N \cup \{0\}$ is onto but *not* one-to-one.
(iii) $n \to |n|^2 : Z \to N \cup \{0\}$ is *neither* one-to-one *nor* onto.

16. $x^4 - 6x^3 + 10x^2 - 3x$

17. (i) $4x^2 - 6x + 1$,
(ii) $2x^2 + 6x - 1$,
(iii) $x^4 + 6x^3 + 14x^2 + 15x + 5$
(iv) $4x - 9$

18. ϕ

19. No. Inclusion is *not* symmetric.

20. 2^{mn}, since $A \times B$ has mn elements and has therefore 2^{mn} subsets.

22. (ii), (iv)

23. (ii)

EXERCISE 2.1

1. (a) $\frac{\pi}{12}$ (b) $\frac{-\pi}{8}$ (c) $\frac{17\pi}{9}$ (d) $\frac{7\pi}{3}$

2. (a) $14°19'$ (nearly) (b)$-114°33'$ (nearly) (c) $420°$ (d) $150°$

3. $\frac{5\pi}{12}$ cm

4. $\frac{20\pi}{3}$ cm

5. 6π

6. $25°12'$

7. (a) $11°27'16''$ (b) $18°19'38''$ (nearly) (c) $29°47'$ (nearly)

8. $100°$

EXERCISE 2.2

1. $\sin\theta = \frac{\sqrt{3}}{2}$, $\tan\theta = -\sqrt{3}$, $\operatorname{cosec}\theta = \frac{2}{\sqrt{3}}$, $\sec\theta = -2$, $\cot\theta = -\frac{1}{\sqrt{3}}$

2. $\cos\theta = \frac{4}{5}$, $\tan\theta = \frac{3}{4}$, $\operatorname{cosec}\theta = \frac{5}{3}$, $\sec\theta = \frac{5}{4}$, $\cot\theta = \frac{4}{3}$

3. $\sin\theta = -\frac{3}{5}$, $\cos\theta = -\frac{4}{5}$, $\operatorname{cosec}\theta = -\frac{5}{3}$, $\sec\theta = -\frac{5}{4}$, $\cot\theta = \frac{4}{3}$

EXERCISE 2.3

1. $\frac{220}{221}$, $\frac{171}{221}$, $\frac{220}{21}$

13. $\frac{2}{\sqrt{5}}$, $\frac{1}{\sqrt{5}}$, 2

14. $\sqrt{\frac{2}{3}}$, $\frac{-1}{\sqrt{3}}$, $-\sqrt{2}$

15. $\sqrt{4+\frac{\sqrt{15}}{8}}$, $\sqrt{4-\frac{\sqrt{15}}{8}}$, $4+\sqrt{15}$

16. $\sqrt{\frac{4-\sqrt{2}-\sqrt{6}}{2\sqrt{2}}}$, $\sqrt{\frac{4+\sqrt{2}+\sqrt{6}}{2\sqrt{2}}}$, $-(\sqrt{2}+1)+\sqrt{4+2\sqrt{2}}$

EXERCISE 2.4

1.	(i) .9387	(ii) .7431	(iii) 1.402	(iv) 1.501
2.	(i) $32°30'$	(ii) $89°30'$	(iii) $88°20'$	(iv) $18°20'$
3.	(i) .5645	(ii) .4295	(iii) .9037	(iv) .9185
4.	(i) $31°36'$	(ii) $87°34'$	(iii) $38°45'$	(iv) $7°49'$

EXERCISE 2.7

1. $\theta = (2n+1)\frac{\pi}{4}$ or $2m\pi$, where $n, m \in I$

2. $\theta = n\pi + (-1)^{n+1}\frac{\pi}{6}$ or $(2m+1)\frac{\pi}{2}$, where $n, m \in I$

3. $\theta = \frac{n\pi}{2}$ or $\frac{m\pi}{2} - \frac{\pi}{8}$ where $n, m \in I$.

4. $\theta = n\pi$ or $2m\pi$, where $n, m \in I$.

5. $\theta = 2n\pi - \frac{\pi}{2}$ or $\frac{2m\pi}{5} - \frac{\pi}{10}$, where $n, m \in I$.

6. $x = 2n\pi + \frac{\pi}{4} \pm A$, where $n \in I$.

7. $\theta = n\pi + (-1)^n \sin^{-1}\left(\frac{-1+\sqrt{17}}{8}\right)$

   or $n\pi + (-1)^n \sin^{-1}\left(\frac{-1-\sqrt{17}}{8}\right)$, where $n \in I$.

8. $\theta = \frac{2k\pi}{m+n}$ or $\frac{(2k+1)\pi}{m-n}$, where $k \in I$.

9. $\theta = n\pi - \frac{\pi}{4}$ or $m\pi + \tan^{-1}\frac{1}{2}$, where $n, m \in I$.

10. $\theta = n\pi + (-1)^{n-1}\frac{\pi}{2}$ or $m\pi + (-1)^{n+1}\frac{\pi}{6}$, where $n, m \in I$.

EXERCISE 4.1

1. (i) $\sqrt{97}$ (ii) $2a\left|\sin\left(\frac{\alpha-\beta}{2}\right)\right|$ 4. 5, −3 5. (3,0) 6. x−y =3

EXERCISE 4.2

1. (a) $\frac{1}{2}$ (b) 18 (c) $\frac{75}{2}$. (d) $\frac{57}{2}$ 3. x=1 4. (7,2) and (1,0) 5. 5x−4y+1 = 0

EXERCISE 4.3

1. (0,0), (3,−9) 3. $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$ 4. (−4, −15)
5. 1 : 3 6. 1 : 2

EXERCISE 4.4

1. (a) Angle of inclination is acute.
   (b) Parallel to the x-axis or coincides with the x-axis.
   (c) Angle of inclination is obtuse

2. (a) 0 (b) 1 (c) undefined (d) $\frac{-1}{\sqrt{3}}$

3. (a) 0 (b) -1 (c) $-\frac{1}{6}$

5. (a) Parallel (b) Neither (c) Perpendicular (d) Parallel

6. 1

7. 9

EXERCISE 4.5

1. $5x + 3y = 9$

2. $x^2 - 8x - 4y + 20 = 0$

3. $y = 3x$

4. $y = x + a$ where a is the given distance

5. $\frac{x^2}{5} + \frac{y^2}{9} = 1$

6. $y = \frac{2}{3}x$

EXERCISE 5.1

1. $y = 2x - 5$

2. $y = -2(2x + 1)$

3. $5y = 3x - 15$

4. $y = -2$

5. $y = x\sqrt{3} + 2$, $y = -x\sqrt{3} + 2$; $y = \pm x\sqrt{3} - 2$; $\left(-\frac{2}{3}\sqrt{3}, 0\right)$, $\left(\frac{2}{3}\sqrt{3}, 0\right)$

6. $30^\circ, 60^\circ$

7. $3y = 2x + 12$

8. $3y = 5(x + 3)$

9. $2x + y = 6$ *or* $x + 2y = 6$

10. $y = 4$, $2x - 3y = 0$, $2x - y = 0$

11. $x = 2$, $7y = 6x + 79$, $7y = -(6x + 65)$

12. $5y = -2x + 18$

13. $3y = 2x - 7$

14. $x - y = 1$

16. (i) $y + x - 3\sqrt{2} = 0$
(ii) $y + \sqrt{3}x - 10 = 0$
(iii) $y - x + 5\sqrt{2} = 0$
(iv) $y = 1$

17. (i) $\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} = \sqrt{2}$
(ii) $\frac{4}{5}x + \frac{3}{5}y = \frac{9}{5}$
(iii) $x = 4.4$ (iv) $y = 2.2$

18. $x - 4y - 7 = 0$

19. $x + y - 5\sqrt{2} = 0$

21. 30°

22. $x = 0$, $\sqrt{3}y - x = 0$, $(0, -3)$, $\left(\frac{-3\sqrt{3}}{2}, \frac{-3}{2}\right)$

23. Coincident 24. Intersecting 25. Parallel 26. $\frac{16}{5}$ 27. 2 28. 1

EXERCISE 6.1

1. (i) $2x + 29y = 0$
(ii) $13x - 19y = 83$
(iii) $x + 12y = 1$
(iv) $3x - 29y = 29$

2. (i) $42x + 21y = 257$
(ii) $21y = 113$
(iii) $7x = 24$
(iv) $63x + 105y = 781$

3. $15x + 12y = 7$ 4. $6x - 17y = -24$

EXERCISE 6.2

1. $x = 2y, x = 3y$ 2. $ax - by = 0$ and $bx + ay = 0$. 3. $\tan^{-1}\frac{1}{3}$

EXERCISE 6.3

1. (i) $x^2 + xy = 0$
   (ii) $xy - y^2 = 0$
   (iii) $xy = 0$
   (iv) $x^2 - y^2 = 0$

6. $2x = 4y + 1, \qquad 2x + y = 0$

MATHEMATICS

A Textbook for Class XI

Part II

IZHAR HUSAIN U. B. TEWARI
M.S. RANGACHARI D. D. JOSHI
V. KANNAN B. DEOKINANDAN

राष्ट्रीय शैक्षिक अनुसंधान और प्रशिक्षण परिषद्
NATIONAL COUNCIL OF EDUCATIONAL RESEARCH AND TRAINING

Contents

Chapters 1-6 have been covered in Part I.
Part III of this textbook will cover Chapters 13 - 19.

CHAPTER 7

Circles and Family of Circles

7.1 Standard Form of the Equation of a Circle

As we have seen before, if we are given a set of points in the plane satisfying some geometrical condition, the coordinates of all the points in the set satisfy a certain relation. If this relation is an equation, it is called the equation of that set of points. We again emphasise that every point of the set must satisfy the equation of the set and no point outside the set should satisfy the equation.

We already know the geometrical condition which will ensure that a set of points in the plane shall form a circle. From this we shall derive the equation of a circle.

Definition 7.1

A circle is the set of all points in a plane which are at a fixed given distance from a fixed point in the plane.

The fixed point is called the centre and the given distance is called the radius of the circle.

Now, we obtain the equation of a circle, as under, with a given centre and radius.

Theorem 7.1

If the centre of the circle is at $C(h,k)$ and radius is r then the equation of the circle is given by

$$(x-h)^2+(y-k)^2=r^2 \tag{7.1}$$

Proof: Let $P(x,y)$ be any point on the circle (See Fig. 7.1). Join the point $P(x,y)$ with the centre $C(h,k)$. Then

$$|CP|=r$$

Using the distance formula, we get

$$|CP|=\sqrt{(x-h)^2+(y-k)^2}$$

Therefore, $\sqrt{(x-h)^2+(y-k)^2}=r$

or $(x-h)^2+(y-k)^2=r^2$

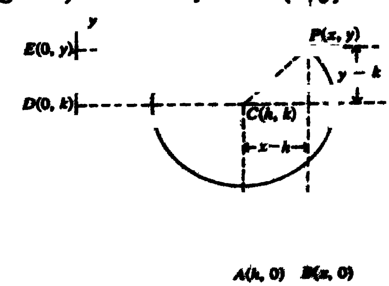


Fig 7.1

Again, if a point $Q(x_1, y_1)$ satisfies the equation (7.1), i.e. if

$$(x_1 - h)^2 + (y_1 - k)^2 = r^2$$

then surely the distance from $Q(x_1, y_1)$ to $C(h, k)$ is r and so Q is on the circle with centre C and radius r. Hence (7.1) is precisely the equation of the circle with centre $C(h, k)$ and radius r.

Corollary: Equation of the circle with centre origin O(0,0) and radius r is

$$x^2 + y^2 = r^2 \tag{7.2}$$

We note that an equation of the type

$$(x - h)^2 + (y - k)^2 = a$$

will represent a circle if $a > 0$; its centre will be (h, k) and radius $\sqrt{a}$. If a=0, the equation is satisfied only by the point (h, k). If $a < 0$, no point in the plane can satisfy the equation.

Example 7.1

Find the equation of the circle with centre $(-3, 2)$ and radius 4.

Solution

Here the point (h, k) is $(-3, 2)$ and radius 4. Hence, substituting $h = -3$, $k = 2$ and $r = 4$ in (7.1), we have

$$[x - (-3)]^2 + (y - 2)^2 = 4^2$$

or

$$(x + 3)^2 + (y - 2)^2 = 16$$

which is the required equation of the circle.

Example 7.2

Find the centre and radius of the circle

$$x^2 + y^2 - 2x + 4y = 8$$

Solution

We write the given equation in the form

$$(x^2 - 2x) + (y^2 + 4y) = 8$$

Now, completing the squares within parentheses, we get

$$\begin{aligned}(x^2 - 2x + 1) + (y^2 + 4y + 4) &= 8 + 1 + 4 \\ (x - 1)^2 + (y + 2)^2 &= 13\end{aligned}$$

Comparing it with the standard form of the equation of the circle, we see that the centre of the circle is $(1, -2)$ and radius is $\sqrt{13}$.

EXERCISE 7.1

1. Find the centre and radius of each of the following circles:

   (i) $x^2 + (y-1)^2 = 2$

   (ii) $(x+5)^2 + (y-3)^2 = 30$

   (iii) $(x - \frac{1}{2})^2 + (y + \frac{1}{3})^2 = \frac{1}{4}$

   (iv) $x^2 + y^2 - 4x + 6y = 5$

   (v) $x^2 + y^2 - x + 2y - 3 = 0$

2. Find the equation of the circle with:

   (i) Centre $(\frac{1}{2}, \frac{1}{4})$ and radius $\frac{1}{12}$

   (ii) Centre $(-3, -2)$ and radius 7

   (iii) Centre $(0, -1)$ and radius 1

   (iv) Centre $(\frac{1}{2}, \frac{1}{2})$ and radius $\frac{1}{2}\sqrt{2}$

   (v) Centre (a, a) and radius $\sqrt{2}a$

   (vi) Centre $(a\cos\alpha, a\sin\alpha)$ and radius -

3. Find the equation of a circle of radius 5 whose centre lies on x-axis and passes through the point (2,3).

7.2 General Form of the Equation of a Circle

As we have seen if a circle has centre (h, k) and radius r, its equation is

$$(x-h)^2 + (y-k)^2 = r^2$$

i.e.

$$x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0$$

Conversely, if we start with an equation of the type

$$x^2 + y^2 + 2gx + 2fy + c = 0 \tag{7.3}$$

where g,f,c are some constants, we can express it as

$$(x+g)^2 + (y+f)^2 = g^2 + f^2 - c$$

and therefore if $g^2 + f^2 - c > 0$, (7.3) will represent the circle whose centre is $(-g, -f)$ and radius is $\sqrt{g^2 + f^2 - c}$.

Thus (7.3) is the General Form of the equation of a circle. It represents a real circle if $g^2 + f^2 - c > 0$, i.e. if $c < g^2 + f^2$. If $c = f^2 + g^2$, the equation represents only the point circle and if $c > g^2 + f^2$, (7.3) does not represent any circle.

Note that an equation of the type

$$ax^2 + ay^2 + 2bx + 2dy + e = 0 \qquad (a \neq 0)$$

can be written as

$$x^2 + y^2 + 2\frac{b}{a}x + 2\frac{d}{a}y + \frac{e}{a} = 0,$$

i.e. in the form (7.3).

If $a \neq b$, then the equation

$$ax^2 + by^2 + 2cx + 2dy + e = 0$$

cannot be expressed in the form (7.3) and hence does not represent a circle.

Remark

The equation of a circle has the following properties:

(i) It is a second degree equation in x and y.

(ii) It contains no term of the form xy.

(iii) Coefficients of x^2 and y^2 are always equal.

To determine the coordinates of the centre of a circle (if the equation of a circle is given) we make the coefficients of x^2 and y^2 equal to 1, and then take the negative halves of the coefficients of x and y. For example, the centre of the circle $x^2+y^2-8x-12y-48=0$ is (4,6) and that of the circle $3x^2+3y^2+12x+18y-12=0$ is $(-2,-3)$.

Since equation (7.3) involves three constants, at least three conditions are required to determine a circle and any given condition gives a relation among all the three constants. However, we may note that having the centre specified is equivalent to two conditions, since its coordinates determine two of the constants in these equations. The three conditions may take various forms. We illustrate some of these in the following examples.

Example 7.3

Find the equation of the circle that passes through the points $(1,0)$, $(-1,0)$, and $(0,1)$.

Solution

Substituting the coordinates of the three points successively in equation (7.3), we get

$$\begin{aligned} 2g + c &= -1 \\ -2g + c &= -1 \\ 2f + c &= -1 \end{aligned}$$

The solution of these simultaneous equations results in $f=0$, $g=0$ and $c=-1$. Therefore, the equation of the circle passing through the three given points is $x^2+y^2=1$.

Example 7.4

Find the equation of the circle which passes through the points (20,3), (19,8) and (2, −9). Find its centre and radius.

Solution

By substitution of coordinates in the general equation of a circle, we have

$$\begin{aligned} 40g + 6f + c &= -409 \\ 38g + 16f + c &= -425 \\ 4g - 18f + c &= -85 \end{aligned}$$

From these three equations, we get

$$g = -7, f = -3 \quad \text{and} \quad c = -111.$$

Hence, the equation of the circle is

$$x^2 + y^2 - 14x - 6y - 111 = 0,$$

or
$$(x-7)^2 + (y-3)^2 = 13^2$$

Its centre is (7,3) and radius is 13.

Example 7.5

Find the equation of the circle concentric with the circle $x^2 + y^2 - 4x - 6y - 9 = 0$ and passing through the point $(-4, -5)$.

Solution

Two circles having same centre are called concentric. Therefore, the centre of the required circle is the centre of the circle $x^2 + y^2 - 4x - 6y - 9 = 0$.

Hence, centre is (2,3).

Now, the circle passes through the points $(-4, -5)$.

Therefore, radius of the circle $= \sqrt{(2+4)^2 + (3+5)^2} = 10$

And equation of the circle is

$$(x-2)^2 + (y-3)^2 = 100$$

or
$$x^2 - 4x + 4 + y^2 - 6y + 9 = 100$$

or
$$x^2 + y^2 - 4x - 6y - 87 = 0$$

EXERCISE 7.2

1. Find the equation of each of the following circles:

   (i) Centre lies on the line $x-4y=1$, and passes through the points (3,7) and (5,5).

   (ii) Passing through the points (2,4) and centre at the intersection of the lines $x-y=4$ and $2x+3y=-7$.

   (iii) Passing through the points $(2,-6)$, (6,4) and $(-3,1)$.

   (iv) Passing through the vertices of the triangle whose sides are along $x+y=2$, $3x-4y=6$, and $x-y=0$.

2. Find the equation of the circle

   (a) passing through (0,0) and intercepting lengths a and b on the axes.

   (b) whose centre is (h,k) and which passes through the point (p,q).

3. Show that the points (5,5),(6,4), $(-2,4)$ and (7,1) all lie on a circle, and find its equation, centre and radius.

4. Find the equation of the circle which passes through the centre of the circle

$$x^2+y^2+8x+10y-7=0$$

and is concentric with the circle

$$2x^2+2y^2-8x-12y-9=0$$

7.3 Equation of a Curve in Parametric Form

Let C be a circle centred at origin and let $P(x,y)$ be any point on it. Let the radius OP of C be r which makes an angle α with the positive direction of x-axis. Draw perpendicular PR from P on x-axis. Then we have

$$x = OR = r\cos\alpha,$$
$$\text{and } y = RP = r\sin\alpha,$$

the coordinates of any point on the circle in terms of the parameter α. The equation of the circle $x^2+y^2=r^2$ is satisfied for all values of α which lie between 0 and 2π for $x=r\cos\alpha$ and $y=r\sin\alpha$.

Thus $x=r\cos\alpha, y=r\sin\alpha, 0\le\alpha<2\pi$, is said to be parametric representation of the circle $x^2+y^2=r^2$ in terms of parameter α.

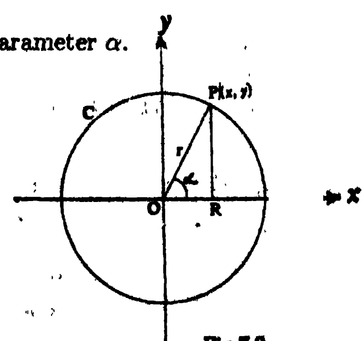


Fig 7.2

Example 7.6
Show that the point

$$x = \frac{2rt}{1+t^2}, \quad y = \frac{r(1-t^2)}{1+t^2} \quad (r \text{ is a constant})$$

lies on a circle for all values of t such that $-1 \leq t \leq 1$. (This is also a parametric equation of a circle.)

Solution
Squaring and adding, we get $x^2 + y^2 = r^2$ which is an equation of the circle.

Example 7.7
Give the parametric form of the circle

$$x^2 + y^2 = 25.$$

Solution
The equation of the circle is given to be

$$x^2 + y^2 = 25$$

Here r=5. Hence the parametric equation in terms of θ is

$$x = 5\cos\theta, \quad y = 5\sin\theta.$$

Example 7.8
Find the cartesian equation of the curve

$$\begin{aligned} x &= 5 + 3\cos\alpha \\ y &= 7 + 3\sin\alpha \end{aligned}$$

Solution
Cartesian equation means an equation connecting x with y directly without involving the parameter α. So we must 'eliminate' α from the given parametric equations.

We write the given equation in the following form:

$$\cos\alpha = \frac{x-5}{3} \quad \text{and} \quad \sin\alpha = \frac{y-7}{3}$$

Since $\sin^2\alpha + \cos^2\alpha = 1$

$$\begin{aligned} \left(\frac{x-5}{3}\right)^2 + \left(\frac{y-7}{3}\right)^2 &= 1 \\ \text{or} \quad (x-5)^2 + (y-7)^2 &= 9 \end{aligned}$$

which is the required equation.

Example 7.9

Find the Cartesian equation of the curve

$$x = at^2$$
$$y = 2at$$

Solution

Here the parameter is t, so we must eliminate t. We find t from the second equation and put it in the first.

$$t = \frac{y}{2a}; \quad x = a\left(\frac{y}{2a}\right)^2 = \frac{y^2}{4a}$$

So the Cartesian equation is $y^2 = 4ax$.

EXERCISE 7.3

1. Find the parametric representation of following circles:

   (i) $3x^2 + 3y^2 = 4$

   (ii) $x^2 + y^2 + 2x - 4y - 4 = 0$

   (iii) $x^2 + y^2 + px + py = 0$

2. Find the equations of the following curves in Cartesian form. Wherever the curve is a circle, find its centre and radius.

   (i) $x = 3\cos\alpha$, $y = 3\sin\alpha$

   (ii) $x = a + c\cos\alpha$, $y = b + c\sin\alpha$

   (iii) $x = 7 + 4\cos\alpha$, $y = -3 + 4\sin\alpha$

   (iv) $x = \frac{t}{2} + 1$, $y = 2t - 1$

7.4 Equation of a Circle when the End Points of a Diameter are Given

Several methods can be given to find the equation of the circle when the coordinates of the end points of any diameter of the circle are given. Here we describe one of them.

Let $P(x, y)$ be any point on the circle of radius r. Let RS be the diameter having coordinates (x_1, y_1) and (x_2, y_2) and let $C(h, k)$ denote the centre of the circle.

Slopes of the straight lines PR and PS are respectively

$$\frac{y-y_1}{x-x_1} \quad \text{and} \quad \frac{y-y_2}{x-x_2}$$

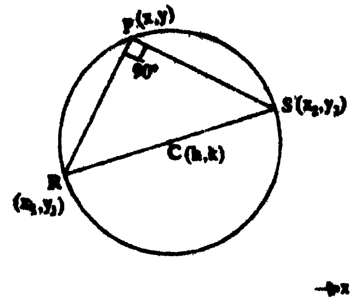


Fig 7.3

Now, since the angle subtended at the point P in the semi-circle RPS is a right angle, therefore, the two lines PR and PS are perpendicular and hence the product of their slopes must be -1 i.e.

$$\frac{y-y_1}{x-x_1} \times \frac{y-y_2}{x-x_2} = -1$$

or
$$(y-y_1)(y-y_2) = -(x-x_1)(x-x_2)$$

or
$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2) = 0 \qquad (7.4)$$

which is the required equation of the circle whose coordinates of the end points of a diameter are (x_1, y_1) and (x_2, y_2).

We can also find the equation by finding the lengths of PR, PS and RS and applying Pythagoras Theorem i.e. in ΔPRS, we have $PR^2 + PS^2 = RS^2$

Further, we can determine the equation of the circle if we find the coordinates of the centre and radius and then we apply the standard form.

Example 7.10

Find the equation of the circle, when the coordinates of the end points of a diameter are (3,4) and $(-3,-4)$.

Solution

Given $x_1 = 3, y_1 = 4; x_2 = -3, y_2 = -4$. Using equation (7.4), we get

$$(x-3)(x+3)+(y-4)(y+4) = 0$$

or
$$x^2 - 9 + y^2 - 16 = 0$$

or
$$x^2 + y^2 = 25$$

which is the required equation of the circle.

EXERCISE 7.4

1. Find the equation of the circle when the coordinates of end points of the diameter are (p,q) and (r,s).
2. Find the equation of the circle drawn on the diagonal of the rectangle as its diameter whose sides are $x = 6, x = -3$; and $y=3$ and $y = -1$.

7.5 Points of Intersection of a Line and a Circle with Centre at Origin, Condition of Tangency

Let the equation of the circle be

$$x^2 + y^2 = r^2 \tag{7.5}$$

and the equation of the line be

$$y = mx + c \tag{7.6}$$

To find the points of intersection of the circle (7.5) and the line (7.6), we have to solve them simultaneously.

Substituting the value of y from equation (7.6) in (7.5), we get

$$x^2 + (mx + c)^2 = r^2$$

or $$x^2 + m^2x^2 + c^2 + 2mxc = r^2$$

or $$(1 + m^2)x^2 + 2mcx + c^2 - r^2 = 0 \tag{7.7}$$

We see that the above equation is quadratic in the variable x and, therefore, it will give two values of x or one value (of course repeating) of x or it will give no value of x. Then we get the corresponding values of y by substituting the values of x obtained from equation (7.7) in equation (7.6).

Thus every line intersects a circle in two points, in one point (in this case the line only touches the circle) or in no point.

Remark

In the last case the line lies somewhere outside the circle, will not touch the circle too.

Now, the equation (7.7) has two distinct roots, one repeated root or no real root according as the discriminant of the equation (7.7) is greater than, equal to or less than zero i.e.

$$4m^2c^2 - 4(1 + m^2)(c^2 - r^2) > 0$$

or $$4m^2c^2 - 4(1 + m^2)(c^2 - r^2) = 0$$

or $$4m^2c^2 - 4(1 + m^2)(c^2 - r^2) < 0$$

i.e. $$4r^2(1 + m^2) > 4c^2$$

$$4r^2(1 + m^2) = 4c^2$$

$$4r^2(1 + m^2) < 4c^2$$

i.e. $$r^2 > \frac{c^2}{1 + m^2} \Longrightarrow r > \left|\frac{c}{\sqrt{1 + m^2}}\right| \tag{7.8}$$

$$r^2 = \frac{c^2}{1 + m^2} \Longrightarrow r = \left|\frac{c}{\sqrt{1 + m^2}}\right| \tag{7.9}$$

$$r^2 < \frac{c^2}{1 + m^2} \Longrightarrow r < \left|\frac{c}{\sqrt{1 + m^2}}\right| \tag{7.10}$$

As the radius is always positive, we take the positive value of r.

Hence, a line intersects a circle in two distinct points, in one point or in no point according as (7.8), (7.9) and (7.10), respectively.

Condition of Tangency

Definition 7.2

A line which meets a circle only in one point is called a tangent to the circle.

(7.9) tells us that $y = mx + c$ is a tangent to the circle $x^2 + y^2 = r^2$ if and only if

$$r = \left|\frac{c}{\sqrt{1+m^2}}\right|, \tag{7.11}$$

i.e. if

$$c = \pm r\sqrt{1+m^2} \tag{7.12}$$

This means that for every value of m, the lines

$$y = mx + r\sqrt{1+m^2} \quad \text{and} \quad y = mx - r\sqrt{1+m^2}$$

are tangent to the circle $x^2 + y^2 = r^2$. These are a pair of parallel tangents to the circle (both have the same slope m). The point where they touch the circle will depend on m.

Example 7.11

Find the coordinates of the point of intersection of the line $y = 1 + x$ and the circle $x^2 + y^2 = 25$.

Solution

The equation of the circle is

$$x^2 + y^2 = 25$$

and the equation of the line is

$$y = 1 + x$$

Substituting this value of y in the equation of the circle, we get

$$x^2 + (x+1)^2 = 25$$

or $$x^2 + x^2 + 1 + 2x = 25$$

or $$2x^2 + 2x - 24 = 0$$

or $$x^2 + x - 12 = 0$$

or $$x^2 - 3x + 4x - 12 = 0$$

or $$x(x-3) + 4(x-3) = 0$$

or $$(x-3)(x+4) = 0$$

from which it follows that $x=3$ or $x=-4$. Putting these values in the equation $y=1+x$, we get the corresponding values of y as $y=4$ or -3. Therefore, the points of intersection are (3,4) and $(-4,-3)$.

Example 7.12

Find the values of p so that the line $3x+4y-p=0$ may be a tangent to the circle

$$x^2+y^2-64=0.$$

Solution

We know that the condition of tangency is

$$c=\pm r\sqrt{1+m^2}$$

The given line is

$$3x+4y-p=0 \quad \text{or} \quad y=-\frac{3}{4}x+\frac{p}{4}$$

Therefore, $c=\frac{p}{4}$ and $m=-\frac{3}{4}$
The radius of the circle is 8.

$$\text{Therefore,}\quad \frac{p}{4}=\pm 8\sqrt{1+\left(\frac{-3}{4}\right)^2}=\pm 8\sqrt{\frac{25}{16}}=\pm\left(8\times\frac{5}{4}\right)=\pm 10$$

$$\text{i.e.}\quad p=\pm 40$$

EXERCISE 7.5

1. Find the coordinates of the point of intersection of the line $7x-y-25=0$ and the circle $x^2+y^2=25$.
2. Find the point of intersection of the line $y=mx+c$ and the circle $x^2+y^2=a^2$

   Also find the condition(s) when the line is tangent to the circle.
3. Find the coordinates of the points of intersection of the line $x+y=2$ with the circle $x^2+y^2=4$.
4. For what value of c will the line $y=2x+c$ be a tangent to the circle $x^2+y^2=5$?
5. Prove that the line $y=mx+r\sqrt{1+m^2}$ touches the circle $x^2+y^2=r^2$ at the point

7.6 Equation of a Tangent to a Circle and Length of the Tangent

Theorem 7.2

The equation of the tangent to the circle $x^2 + y^2 = r^2$ at a point (x_1, y_1) on the circle is given by

$$xx_1 + yy_1 = r^2 \tag{7.13}$$

Proof

As (x_1, y_1) is on the circle,

$$x_1^2 + y_1^2 = r^2 \tag{7.14}$$

Let us take a line through the point (x_1, y_1). If it has slope m, then its equation is

$$y - y_1 = m(x - x_1) \tag{7.15}$$

$$\text{or} \qquad y = mx + (y_1 - mx_1)$$

As we know, the condition that this line is a tangent to the circle $x^2 + y^2 = r^2$ is that

$$y_1 - mx_1 = \pm r\sqrt{1 + m^2} \tag{7.16}$$

$$\begin{aligned} \text{i.e.} \qquad (y_1 - mx_1)^2 &= (1 + m^2)r^2 \\ &= (1 + m^2)(x_1^2 + y_1^2) \end{aligned}$$

In view of (7.14), simplification yields

$$\begin{aligned} x_1^2 + 2x_1my_1 + m^2y_1^2 &= 0 && (7.17) \\ \text{or} \qquad (my_1 + x_1)^2 &= 0 \\ \text{i.e.} \qquad my_1 + x_1 &= 0 \\ \text{i.e.} \qquad \left(\frac{y - y_1}{x - x_1}\right)y_1 + x_1 &= 0 && [\text{from } 7.15] \\ \text{i.e.} \qquad (y - y_1)y_1 + (x - x_1)x_1 &= 0 \\ \text{i.e.} \qquad xx_1 + yy_1 - x_1^2 - y_1^2 &= 0 \\ \text{i.e.} \qquad xx_1 + yy_1 - r^2 &= 0 && (7.18) \end{aligned}$$

Thus, the above equation is the equation of the tangent to the circle $x^2 + y^2 = r^2$ at (x_1, y_1) on it.

Remark

Note that the radius through (x_1, y_1) of the circle $x^2+y^2=r^2$ has the equation

$$\frac{y}{x} = \frac{y_1}{x_1}$$

$$\text{i.e.} \quad y = \frac{y_1}{x_1}x$$

The slope of the tangent to the circle at (x_1, y_1) is $-\dfrac{x_1}{y_1}$ as is seen from (7.18). So the tangent at (x_1, y_1) is perpendicular to the radius through (x_1, y_1). Since any circle can be reduced to the form $x^2+y^2=r^2$ by shifting the origin (see §5.13) to its centre, we conclude that a tangent to a circle is perpendicular to the radius through its point of contact.

Another approach

Let C be a curve. Let P and Q be any two points on the curve C as shown in Fig. 7.4. Draw the line P. Now, as the point Q moves along the curve towards P, the line PQ turns about P (rotates about P), and Q approaches P, the line PQ coincides with the limiting line PT. This limiting line PT is called the *tangent line* to the curve C at the point P.

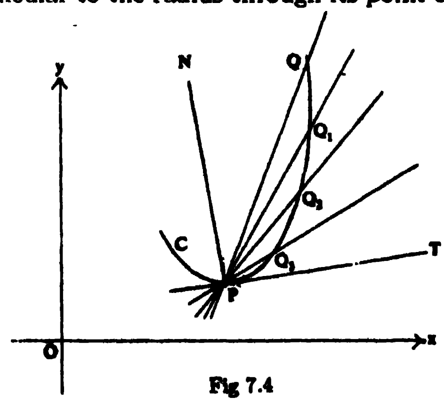


Fig 7.4

The line perpendicular to the tangent to curve C is called the *normal* to the curve C at the point.

Now we will show again that the equation of the tangent to the circle $x^2+y^2=r^2$ at a point (x_1, y_1) is given by

$$xx_1 + yy_1 = r^2 \tag{7.19}$$

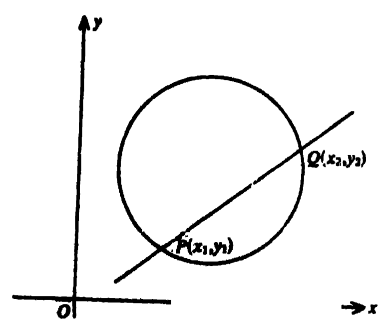


Fig 7.5

Proof

Let (x_1, y_1) be the coordinates of a given point on the circle $x^2+y^2=r^2$. Then this point must satisfy the equation, i.e.

$$x_1^2 + y_1^2 = r^2 \tag{7.20}$$

Take another point $Q(x_2, y_2)$ on the circle. Then

$$x_2^2 + y_2^2 = r^2 \tag{7.21}$$

Now, the equation of PQ by the two-point formula will be

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \tag{7.22}$$

Subtracting (7.21) from (7.20), we have

$$y_1^2 - y_2^2 = x_2^2 - x_1^2$$

or

$$(y_1 - y_2)(y_1 + y_2) = (x_2 - x_1)(x_2 + x_1)$$

Therefore,

$$\frac{y_2 - y_1}{x_2 - x_1} = -\frac{x_2 + x_1}{y_2 + y_1} \tag{7.23}$$

Since the tangent to the curve is the limiting position of the chord PQ as Q approaches P,i.e. (x_2, y_2) approaches (x_1, y_1), it can be shown that x_2 approaches x_1 and y_2 approaches y_1. So

$$\frac{y_2 - y_1}{x_2 - x_1} \quad \text{approaches} \quad -\frac{2x_1}{2y_1} = -\frac{x_1}{y_1}$$

and equation (7.22) reduces to

$$y - y_1 = -\frac{x_1}{y_1}(x - x_1)$$

or

$$xx_1 - x_1^2 + yy_1 - y_1^2 = 0$$

or

$$xx_1 + yy_1 = x_1^2 + y_1^2$$

From (7.20), $x_1^2 + y_1^2 = r^2$. Hence, the above equation gives

$$xx_1 + yy_1 = r^2 \tag{7.24}$$

which is the required equation of the tangent to the circle $x^2 + y^2 = r^2$ at the given point (x_1, y_1).

If the tangent is desired at any point, say (x_1, y_1), of the circle given by the general equation,

$$x^2 + y^2 + 2gx + 2fy + c = 0 \tag{7.25}$$

we can proceed similarly as above. The equation of the tangent will be

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0 \tag{7.26}$$

[The proof is left to the reader].

Remark

It may be noted that the equation of the tangent at (x_1, y_1) is, therefore, obtained from that of the equation of the circle itself by substituting xx_1 for x^2, yy_1 for y^2, $\frac{x + x_1}{2}$ for x and $\frac{y + y_1}{2}$ for y, respectively.

Example 7.13
Find the equation of the tangent to the circle $x^2 + y^2 = 13$ at the point (2,3).

Solution
The equation of the tangent to the circle $x^2 + y^2 = a^2$ at (x_1, y_1).

is

$$xx_1 + yy_1 = a^2$$

Substituting $x_1=2$, $y_1=3$ and $a^2=13$, the equation of the tangent to the given circle is

$$2x + 3y = 13$$

Example 7.14
Find the equation of the tangent to the circle

$$x^2 + y^2 - 26x + 12y + 105 = 0$$

at the point (7,2).

Solution
Comparing the above equation of the circle with the standard equation of the circle, we have

$$2g = -26, 2f = 12, c = 105$$

Also, $x_1 = 7, y_1 = 2$

Therefore, from (7.26), the equation of the tangent is

$$\begin{aligned}
7x + 2y - \frac{26}{2}(x+7) + \frac{12}{2}(y+2) + 105 &= 0\\
\text{or} \quad 7x + 2y - 13(x+7) + 6(y+2) + 105 &= 0\\
\text{or} \quad 7x + 2y - 13x - 91 + 6y + 12 + 105 &= 0\\
\text{or} \quad -6x + 8y + 26 &= 0\\
\text{or} \quad 3x - 4y - 13 &= 0
\end{aligned}$$

Length of the Tangent

Let (x_1, y_1) be the given point P, and let the equation of the given circle be $x^2 + y^2 = r^2$
Then, by Remark in §7.6 of the tangent line

$$\angle OTP = \frac{\pi}{2}$$

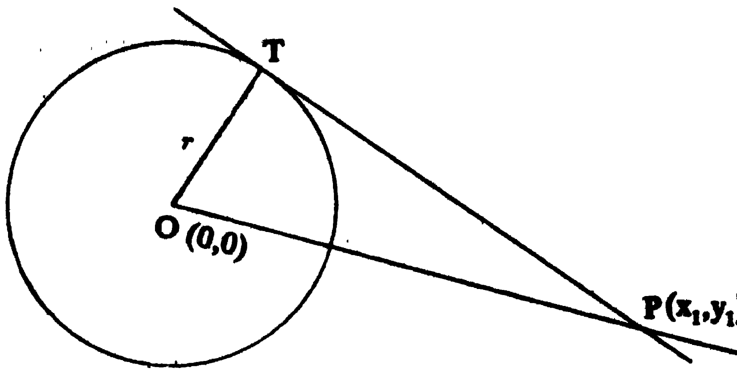


Fig 7.6

Therefore, applying Pythagoras Theorem, we have

$$PT^2 = OP^2 - r^2$$

or

$$\begin{aligned} PT^2 &= (x_1 - 0)^2 + (y_1 - 0)^2 - r^2 \\ &= x_1^2 + y_1^2 - r^2 \end{aligned}$$

Therefore,

$$PT = \sqrt{x_1^2 + y_1^2 - r^2}$$

If the length of the tangent is desired from the point (x_1, y_1) to the circle given by the general equation

$$x^2 + y^2 + 2gx + 2fy + c = 0,$$

we can proceed in the similar way as in the above case. The centre of the circle is $(-g, -f)$ and radius is $r = \sqrt{g^2 + f^2 - c}$. Therefore, applying again Pythagoras Theorem, we get

$$\begin{aligned} PT^2 &= (x_1 + g)^2 + (y_1 + f)^2 - (g^2 + f^2 - c) \\ &= x_1^2 + g^2 + 2gx_1 + y_1^2 + f^2 + 2y_1 f - (g^2 + f^2 - c) \\ &= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c \end{aligned}$$

Hence, the length of the tangent PT is given by

$$\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$$

PT^2 is called the *Power of a Point* (x_1, y_1) with respect to the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

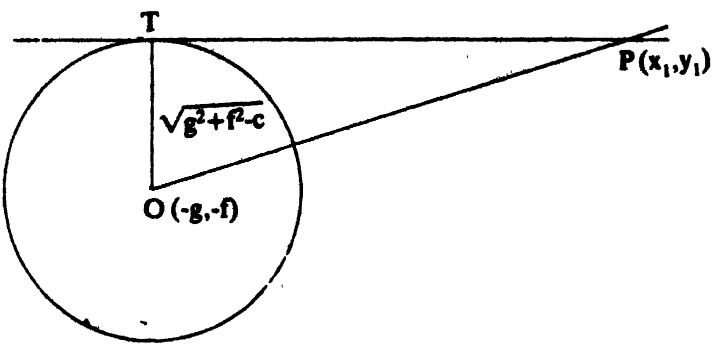


Fig 7.7

Example 7.15
Find the length of the tangent drawn from (3,4) to the circle

$$x^2 + y^2 - 4x + 6y - 1 = 0$$

Solution

Substituting the coordinates in the given equation we get the length of the tangent

$$\begin{aligned} &= \sqrt{3^2 + 4^2 - 4 \times 3 + 6 \times 4 - 1} \\ &= \sqrt{9 + 16 - 12 + 24 - 1} \\ &= \sqrt{36} = 6 \end{aligned}$$

Hence, the length of the required tangent is 6.

EXERCISE 7.6

1. Find the equation of the tangent to the given circles at the indicated points:

(i) $x^2 + y^2 = 2$; (1,1)
(ii) $x^2 + y^2 = 10$; (1,−3)
(iii) $x^2 + y^2 = 25$; (−3,−4)
(iv) $x^2 + y^2 - 30x + 6y + 109 = 0$; (4,−1)
(v) $x^2 + y^2 - 26x - 2y + 45 = 0$; (2,3)
(vi) $x^2 + y^2 - 2x - 10y + 1 = 0$; (−3,2)

(vii) $x^2 + y^2 - 2ax = 0$; $(a(1 + \cos\alpha), a\sin\alpha)$

2. Show that the point $(a\cos\alpha, a\sin\alpha)$ lies on the circle $x^2 + y^2 = a^2$ for all values of α. Also show that the equation of the tangent at this point is $x\cos\alpha + y\sin\alpha = a$. Further show that the line $x\sin\alpha - y\cos\alpha = a$ is a tangent to the circle for all values of α.
3. Find the condition that the line $lx + my = n$ is a tangent to the circle $x^2 + y^2 = a^2$.
4. Find the equations of the tangent lines to the circle $x^2 + y^2 = 9$ which are parallel to the line $2x + y - 3 = 0$.
5. Find the length of the tangent drawn from the point $(-2,3)$ to the circle $2x^2+2y^2 = 3$.
6. Find the length of the tangent drawn from the point $(-2,-3)$ to the circle $x^2 + y^2 - 2x - 10y + 1 = 0$.
7. Prove that the tangents to the circle $x^2 + y^2 = 169$ at $(5,12)$ and $(12,-5)$ are perpendicular to each other.
8. Find the equation of the two tangents to $x^2 + y^2 = 3$ which makes an angle of $60°$ with the axis of x.
9. Show that the line $x + y\sqrt{3} = 4$ touches the circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 4x - 4\sqrt{3}y + 12 = 0$ at the same point.

7.7 Families of Circles through the Intersection of Two Circles

We have seen that there are three unknown constants in the general equation of a circle given by

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Thus we require three conditions to determine a circle. If we impose two conditions on a circle, one of the constants in its equation will be undetermined. This condition may be chosen arbitrarily, provided the condition is independent of and does not contradict the conditions which determine the other two. We have, therefore, a system in which only one arbitrary constant appears; consequently, we have a one-parameter family of circles. Thus a one-parameter family of circles is formed in the same way as the imposition of one condition on a straight line leads to a one-parameter family of straight lines.

The following examples will illustrate what we have said in the above development.

Example 7.16

Write the equation of the family of circles with centre at $(-4,3)$.

Solution

Since having the centre specified is equivalent to imposing two conditions on the circle, we can determine two constants, namely, h and k in the general equation of the circle given by $x^2 + y^2 + 2gx + 2fy + c = 0$.

Also since $g=4$ and $f=-3$, from the above equation, we have

$$x^2+y^2+8x-6y+c=0$$

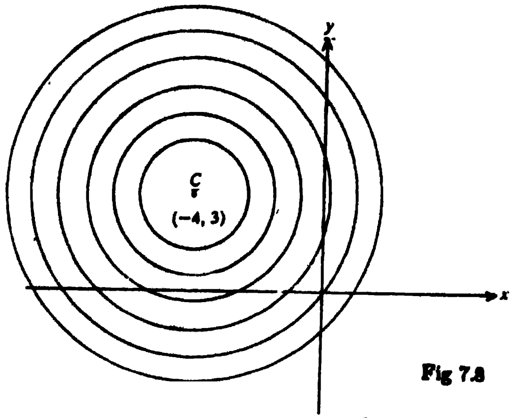


Fig 7.8

In this equation c is the undetermined constant, that is, a parameter. We thus get a one-parameter family of circles. Fig.7.8 shows some members of this family for various values of c.

A family of circles with the same centre and different radii is called a *Family of Concentric Circles.* The family of circles in the above example is a Family of Concentric Circles.

Example 7.17

Find an equation of the circle which passes through the points $(-2,2)$ and $(5,1)$ and has its centre on the line $x+2y+3=0$.

Solution

Let (h,k) be the centre of the circle. Since the centre lies on the line $x+2y+3=0$, we have

$$\begin{aligned} h+2k+3 &= 0 \\ \text{or} \qquad h &= -(2k+3) \end{aligned}$$

Then the family of all circles with centres on the line may be written as

$$x^2+y^2+2(2k+3)x-2ky+(2k+3)^2+k^2-r^2=0$$

$$\text{or } x^2+y^2+2(2k+3)x-2ky+c=0, \quad \text{where} \quad c=(2k+3)^2+(k^2-r^2) \qquad (7.27)$$

Thus, we get a two-parameter family of circles since there are two arbitrary constants k and c left in the equation. Let us now impose the remaining two conditions, namely, that the circle passes through the points $(-2,2)$ and $(5,1)$. We get

$$-12k + c = 4 \quad \text{and} \quad 18k + c = -56$$

Thus $c = -20$ and $k = -2$. Substituting these values in (7.27), we get the equation of the circle as

$$x^2 + y^2 - 2x + 4y - 20 = 0.$$

Now we will discuss the family of circles through the intersection of two circles. Let

$$S_1 = x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0 \tag{7.28}$$

$$\text{and} \quad S_2 = x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0 \tag{7.29}$$

be the equations of two circles. Now consider the equation

$$S_1 + kS_2 = 0$$

$$\text{i.e.} \quad x^2 + y^2 + 2g_1x + 2f_1y + c_1 + k(x^2 + y^2 + 2g_2x + 2f_2y + c_2) = 0 \tag{7.30}$$

where k is an arbitrary constant. This equation may be written in the form

$$(1 + k)x^2 + (1 + k)y^2 + 2(g_1 + kg_2)x + 2(f_1 + kf_2)y + (c_1 + kc_2) = 0 \tag{7.31}$$

This is clearly the equation of a circle if $k \neq -1$. If $k = -1$, (7.31) reduces to the equation of a straight line given by

$$S_1 - S_2 = 0$$

$$\text{i.e.} \quad 2(g_1 - g_2)x + 2(f_1 - f_2)y + (c_1 - c_1) = 0$$

In either case, if the two circles have a point of intersection, say, (x_1, y_1) the curve given by (7.31) will pass through this point. This is so because if (x_1, y_1) satisfies the equations of the two circles, then substitutions of these coordinates in (7.30) reduces it to

$$0 + k(0) = 0.$$

We say that (7.30) or (7.31), for $k \neq -1$, represents a one-parameter family of circles, all members of which pass through the points of intersection of two given circles, if they have any. If $k = -1$, (7.30) and (7.31) reduce to the equation of a straight line which passes through the intersection of the two given circles, if they have any.

Example 7.18

Find the equation of the circle through the intersection of the two circles

$$x^2 + y^2 - 8x - 2y + 7 = 0$$
$$x^2 + y^2 - 4x + 10y + 8 = 0$$

and that passes through the point $(-1, -2)$.

Solution

The equation of the family of circles through the intersection of the two given circles is

$$(x^2 + y^2 - 8x - 2y + 7) + k(x^2 + y^2 - 4x + 10y + 8) = 0$$

Since the circle is one member of this family and it passes through the point $(-1, -2)$, substituting these coordinates in the above equation, we get

$$24 - 3k = 0 \quad \text{or,} \quad k = 8.$$

This value of k gives the required equation as

$$9x^2 + 9y^2 - 40x + 78y + 71 = 0.$$

EXERCISE 7.7

Find the equation of the family of circles described in each of the questions 1 to 3, and select the specified member.

1. Centre at $(2, -1)$; member with radius 3.
2. Centre at $(-4, 2)$; member tangent to $x - y = 3$
3. Centre at $(3, -5)$; member through $(0, -1)$
4. Find the equation of the circle through the intersection of the two circles

$$x^2 + y^2 - 8x - 2y + 7 = 0$$
$$\text{and} \quad x^2 + y^2 - 4x + 10y + 8 = 0$$

and which satisfies the following additional condition:

(i) has its centre on the x-axis,

(ii) passes through the point $(3, -3)$.

5. Find the equations of the circles which have centres on $2x - 3y + 4 = 0$.
6. Show that the equation $x^2 + y^2 - 2hx - 2hy + h^2 = 0$ represents the family of circles which touch both the coordinate axes.

7.8 Condition for Two Intersecting Circles to be Orthogonal

Before we find the condition for two circles to be orthogonal, we define the angle between two curves and their orthogonality.

Definition 7.3

The angle between two curves at their point of intersection is the angle between the tangents to the curves at that point of intersection.

Definition 7.4

Two curves are said to intersect orthogonally when the tangents at the common point are at right angles.

Now we find the condition of orthogonality for two circles.

$$\text{Let} \qquad x^2 + y^2 + 2gx + 2fy + c = 0 \tag{7.32}$$

$$\text{and} \qquad x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0 \tag{7.33}$$

be the equations of two given circles. The centres of the circles are $(-g, -f)$ and $(-g_1, -f_1)$ and their respective radii are $\sqrt{g^2 + f^2 - c}$ and $\sqrt{g_1^2 + f_1^2 - c_1}$.

Let PT and PT_1 be the tangents to the circles (7.32) and (7.33), respectively, at the intersecting point P. Now, since as we know that the radius of a circle through a point is perpendicular to the tangent at that point, it easily follows that the angle of intersection of the circles is same as the angle between their radii drawn through

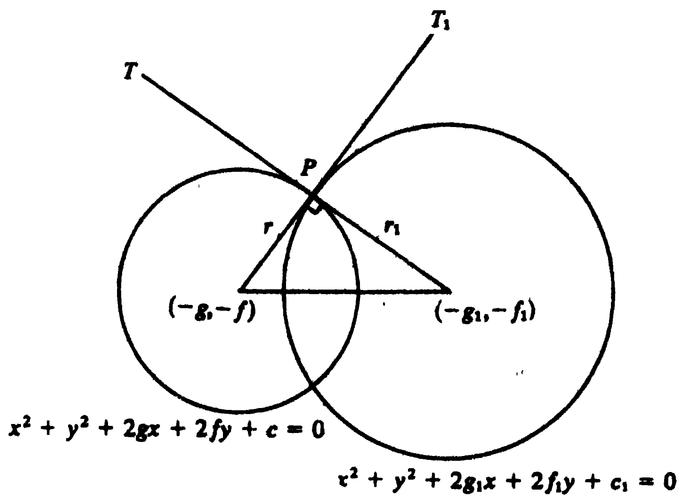


Fig 7.9

the points of intersection. In the case of orthogonal intersection, this angle is $\frac{\pi}{2}$ and consequently the square of the distance between the centres of the circles is equal to the sum of the squares of their radii.

For orthogonal intersection we then have

$$(g - g_1)^2 + (f - f_1)^2 = g^2 + f^2 - c + g_1^2 + f_1^2 - c_1$$

or
$$g^2 + g_1^2 - 2gg_1 + f^2 + f_1^2 - 2ff_1 = g^2 + f^2 - c + g_1^2 + f_1^2 - c_1$$

or
$$2gg_1 + 2ff_1 = c_1 + c_7 \tag{7.34}$$

This is the required condition.

Example 7.19

Find the equation of the circle which cuts orthogonally the circle $x^2+y^2-6x+4y-3=0$, passes through (3,0) and touches the axis of y.

Solution

The equation of the circle which touches the axis of y is

$$x^2 + y^2 - 2hx - 2ky + k^2 = 0$$

If it passes through (3,0), then we have

$$k^2 = 6h - 9 \qquad (7.35$$

If it cuts the given circle orthogonally,

$$6h - 4k = k^2 - 3 \qquad (7.36$$

From (7.35) and (7.36), k=3 and h=3.
Therefore, the required equation is

$$x^2 + y^2 - 6x - 6y + 9 = 0.$$

EXERCISE 7.8

1. Find the equation to the circle which passes through (1,1) and cuts orthogonally each of the circles $x^2 + y^2 - 8x - 2y + 16 = 0$ and $x^2 + y^2 - 4x - 4y - 1 = 0$.
2. Prove that the two circles which pass through two points $(0,a)$ and $(0,-a)$ and touch the straight line $y = mx + c$ will cut orthogonally if $c^2 = a^2(2 + m^2)$.
3. Let the equation of two circles whose radii are r,r' be $S = 0$, $S' = 0$; then show that the circles $\frac{S}{r} \pm \frac{S'}{r'} = 0$ will intersect at right angles.

CHAPTER 8

Conic Sections

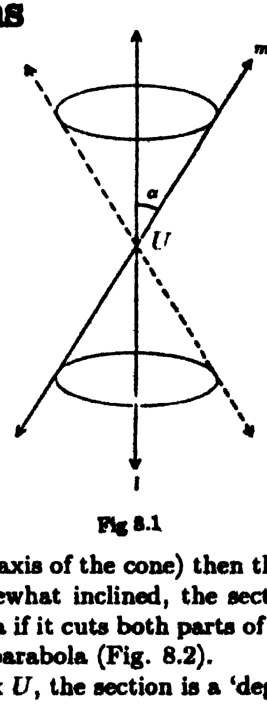


Fig 8.1

8.1 Sections of a Cone

Suppose we take a vertical line l and another line m inclined to l at an angle α and intersecting it at the point U.
Suppose we rotate m around the line l (so that in all positions of m, the angle α remains constant). Then the surface generated is a cone (or a double cone) with vertex U and the line l as its axis. Any position of line m is called a generator of this cone.
If we take a plane then the points common to the plane and the cone form the section of the cone by the plane.
If the plane is horizontal (that is, perpendicular to the axis of the cone) then the section of the cone is a circle (Fig. 8.1). If the plane is somewhat inclined, the section is an ellipse if it cuts only one part of the cone or a hyperbola if it cuts both parts of the cone. If the plane is parallel to a generator, the section is a parabola (Fig. 8.2).

In case the cutting plane passes through the vertex U, the section is a 'degenerated conic', that is, a point (a degenerate circle or ellipse), a single straight line (a degenerate parabola) or a pair of straight lines (a degenerate hyperbola).

For this reason, circle, ellipse, parabola hyperbola are called conic sections.

Many important discoveries, both in mathematics and science, have been linked to the conic sections. The Greeks, particularly Archimedes and Apollonius studied these curves for their own beauty. The first application appeared almost 2000 years later when, about the year 1604, Galileo discovered that if a projectile is fired horizontally from the top of a tower, then its path would be a parabola. Only a few years later, in 1609, Kepler discovered that the orbit of Mars is an ellipse and went on to suggest that all planets move in elliptic paths.

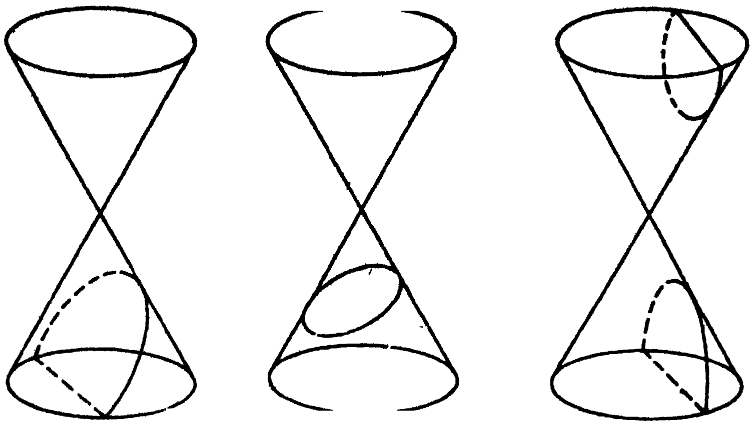


Fig. 8.2

Indeed these curves are important tools for present-day exploration of outer space and also for research into the behaviour of atomic particles

We shall be studying the conic sections as plane curves. For this purpose it is convenient to use equivalent definitions that refer only to the plane in which the curve lies, and refer to special points and lines in this plane called *foci* (focus is the singular) and *directrix*.

According to this approach, parabola, ellipse and hyperbola are defined in terms of a fixed point (called focus) and a fixed line (called directrix) in the plane.

If S is the focus and l is the directrix, then the set of all points in the plane whose distance from S bears a constant ratio e to their distance from l is a conic section. If $e < 1$, the conic section is an ellipse, if $e = 1$, it is a parabola and if $e > 1$, it is a hyperbola. The constant e is called the eccentricity of the conic section.

There are simple and elegant arguments which show that these local properties of ellipse, hyperbola and parabola follow from their corresponding definitions as section of a cone. But we shall not go into them here as they involve concepts from three- dimensional geometry.

In what follows, we shall obtain the equations in standard forms of these conic sections and study them in greater detail.

8.2 Parabola

Definition 8.1

A ***parabola*** is the set of all points whose distances from a fixed point in the plane are equal to their distances from a fixed line in the plane. The fixed point is called the *focus* and the fixed line the *directrix*.

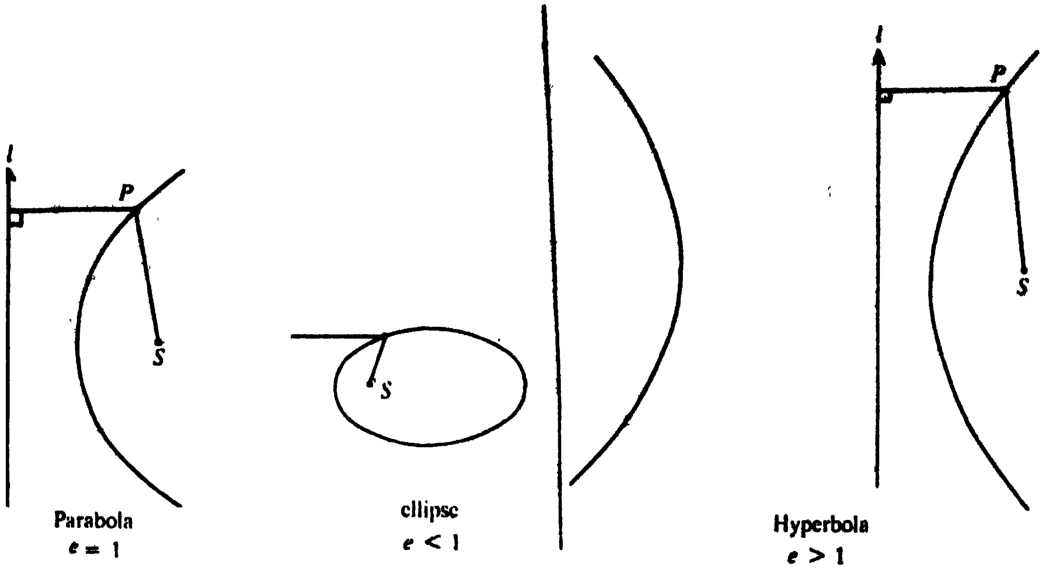


Fig. 8.3

To find the equation of a parabola we set up a coordinate system and place the directrix l and the focus F at convenient places. We place the focus at $(a,0)(a>0)$ and let the directrix l be $x=-a$ as shown in Fig. 8.4. If $P(x,y)$ is a typical point on the parabola, then by using the distance formula the defining condition can be written as:

$$(x-a)^2+y^2=(x+a)^2$$

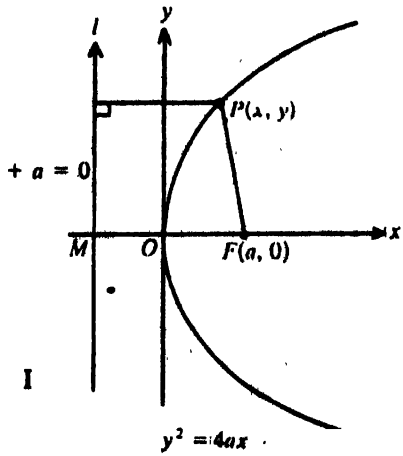


Fig 8.4

Simplifying, we get

$$\begin{aligned} x^2+y^2-2ax+a^2 &= x^2+2ax+a^2 \\ \text{or} \qquad y^2 &= 4ax, \quad a>0 \end{aligned} \tag{8.1}$$

Each point on the locus satisfies this equation. Again, by reversing the steps it can be shown that every point which satisfies this equation fulfills the defining condition. This is, therefore, the equation of the parabola with focus at $(a,0)$ and directrix $x=-a$, $a>0$.

Choosing the focus at $(a,0)$ and the directrix as the line $x=-a$ may appear very artificial and arbitrary. Actually, what we have done is to choose our origin and the directions of the axes so conveniently that the focus turns out to be on the x-axis, so its coordinates must be $(a,0)$ for some a and the directrix would be parallel to the y-axis with its equation $x=-a$. This is done as follows.

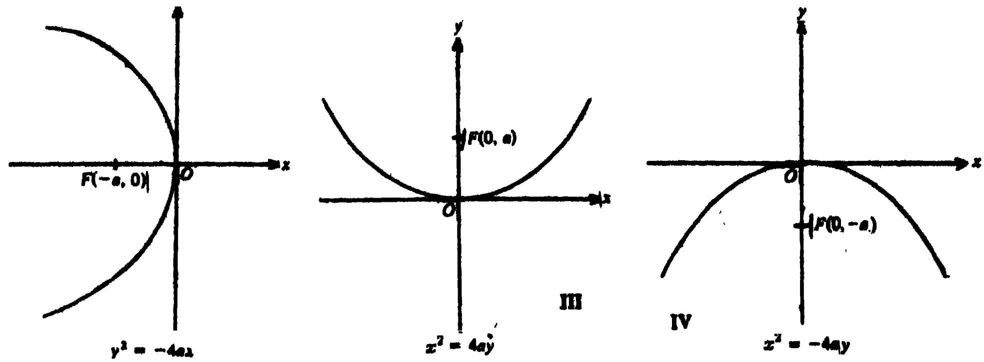


Fig. 8.5

If F is the focus and l the directrix, let M be the foot of the perpendicular from F on l. We take the mid-point O of FM as origin and the direction OF as the positive direction of x-axis. If $OF = a$, coordinates of F are $(a, 0)$. Also as l is perpendicular to line FM, i.e. to x-axis, it is parallel to y-axis. As O is the mid-point of FM, so $OM = OF = a$. Hence equation of directrix is $x = -a$.

If we had not conveniently chosen the origin and the directions of axes, we would still get the equation of the parabola but it would not be so simple as (8.1).

The line through the focus and perpendicular to the directrix is called the *axis* of the parabola and the point where the parabola intersects the axis is called the *vertex*. For the parabola (8.1), the axis is clearly the x-axis and the vertex is at the origin as is easily verified. If we replace y by $-y$, the equation (8.1) remains unchanged. Hence, the x-axis is the line of symmetry for this parabola. In fact the axis of the parabola is its line of symmetry.

If we change the position of the parabola relative to the coordinate axes, we naturally change its equation. Three other simple positions, each with its corresponding equation, are shown in Fig. 8.5. We shall call them the second, third and fourth standard forms of the equation of a parabola. Students should verify the correctness of all three equations. It should be emphasised that the constant a is always understood to be positive. Geometrically, it is the distance from the focus to the vertex, and also from the vertex to the directrix.

Example 8.1

Show that the equation $y^2 - 8y - x + 19 = 0$ represents a parabola. Find its vertex, focus and directrix.

Solution

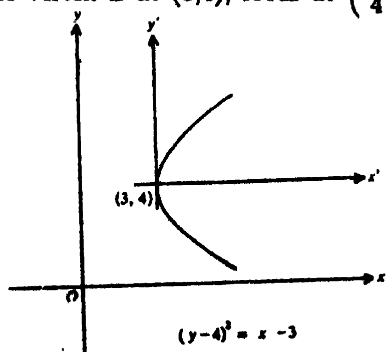
Completing the square, we get

$$(y-4)^2 = x-3$$

If we introduce new variables $x' = x-3$, $y' = y-4$, the equation becomes $y'^2 = x'$ which can be written as $y'^2 = 4.\left(\frac{1}{4}\right)x'$. Thus in the new coordinate system, the vertex is at origin, focus at $\left(\frac{1}{4}, 0\right)$ and directrix $x' + \frac{1}{4} = 0$.

Hence, in the original coordinate system, the vertex is at (3,4), focus at $\left(\frac{13}{4}, 4\right)$ and directrix $x = \frac{11}{4}$

Note:: In exactly the same way see that any equation of the form $y^2 + Ax + By + C = 0$, $A \neq 0$ represents a parabola in the standard form I or II (see Fig. 8.4 and 8.5). Similarly, $x^2 + Ax + By + C = 0$, $B \neq 0$ will represent a parabola in standard form III or IV.



$(y-4)^2 = x-3$

Fig 8.6

EXERCISE 8.1

1. For the following parabolas find the coordinates of the foci and the equations of the directrices:

   (a) $y^2 = 8x$
   (b) $x^2 = 6y$
   (c) $y^2 = -12x$
   (d) $x^2 = -16y$

2. Find the equation of the parabola with vertex at the origin and satisfying the condition:

   (a) Focus at $(-a, 0)$
   (b) Directrix $y = 2$
   (c) Passing through (2,3) and axis along x-axis

3. Find the foci, vertices, directrices and axes of the following parabolas. Also draw their rough sketches.

   (a) $y = x^2 - 2x + 3$

(b) $y = -4x^2 + 3x$

(c) $x^2 + 2y - 3x + 5 =$

4. Find the equation of each of the following parabolas:

(a) Directrix $x = 0$, focus at $(6, 0)$

(b) Vertex at $(0, 4)$, focus at $(0, 2)$

(c) Focus at $(-1, -2)$, directrix $x - 2y + 3 = 0$

8.3 Ellipse

Definition 8.2

An *ellipse* is the set of all points in the plane whose distances from a fixed point in the plane bears a constant ratio, less than one, to their distances from a fixed line in the plane. The fixed point is called focus, the fixed line a directrix and the constant ratio (denoted by e) the eccentricity of the ellipse.

Suppose S is a fixed point and l a fixed line in plane. Let e be a constant such that $0 < e < 1$. Then the set of all points P in the plane such that

$$\frac{PS}{\text{distance of } P \text{ from } l} = e$$

is an ellipse whose eccentricity is e.

Before we obtain the equation of an ellipse, we must make some preparations.

Let the fixed point be S and the fixed line be l. Let Z be the foot of the perpendicular from S to l.

The line ZS will be bisected in the ratio $1 : e$ internally at some point A and externally at some other point A'. As $e < 1$, A' will be beyond S, i.e. S will be between Z and A'. That is

$$\frac{ZA}{AS} = \frac{1}{e}, \quad \frac{ZA'}{A'S} = \frac{1}{e}$$

or $$AS = e.\,AZ \tag{8.2}$$

and $$A'S = e.\,A'Z \tag{8.3}$$

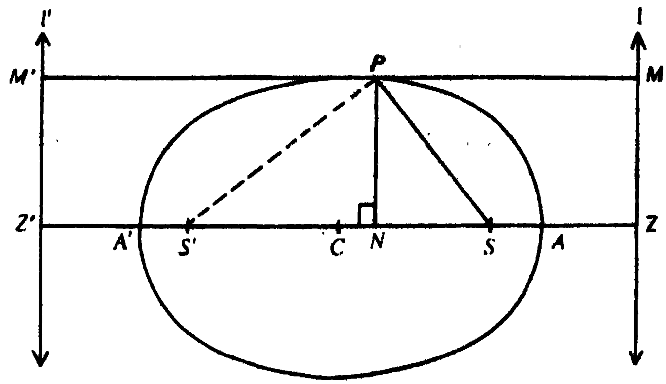


Fig 8.7

Let $AA' = 2a$ and let C be the mid-point of AA'.
Adding (8.2) and (8.3), we have

$$(AS + A'S) = e(AZ + A'Z)$$

i.e.

$$\begin{aligned} 2a &= e(CZ - CA + CZ + A'C) \\ &= 2e.CZ \end{aligned}$$

Therefore,

$$CZ = a/e \tag{8.4}$$

On the other hand, subtracting (8.2) from (8.3)

$$A'S - AS = e(A'Z - AZ),$$

i.e.

$$(CA' + CS) - (CA - CS) = 2ae$$

or

$$2\,CS = 2ae$$

so

$$CS = ae \tag{8.5}$$

Just as A and A' are symmetrically situated with respect to C, we take points S' and Z' symmetrically situated to S and Z respectively with respect to C. That is C is the mid-point of SS' and also of ZZ'. It follows that

$$CS' = ae, \;\; CZ' = \frac{a}{e}$$

We draw the line l' perpendicular to $Z'S$ through Z'.

It is clear from the definition of the ellipse and our choice of A and A', that A and A' lie on the ellipse. As we have said before, S is a focus and l a directrix of the ellipse.

Let $P(x, y)$ be any point on the ellipse. Let PM, PM' and PN be perpendiculars from P to l, l' and CS, respectively. Our first objective is to prove that S' and l' also

form a set of focus and directrix for the same ellipse, i.e.

$$\frac{PS'}{PM'} = e$$

Our second objective is to prove that the sum of the distances of all points on the ellipse from the foci (S and S') is a constant.

$$\begin{aligned} \text{Now} \quad PS &= e.PM \\ \text{so} \quad PS^2 &= e^2 PM^2 \\ \text{i.e.} \quad PN^2 + NS^2 &= e^2.ZN^2 \end{aligned}$$

If N lies on the same side of C as S lies, as shown in Fig. 8.7, then

$$\begin{aligned} (CS - CN)^2 + PN^2 &= e^2(CZ - CN)^2 \\ \text{i.e.} \quad (CS' - CN)^2 + PN^2 &= e^2(CZ' - CN)^2 \end{aligned}$$

$$\begin{aligned} \text{or} \quad (CS' + CN)^2 - 4CS'.CN + PN^2 &= e^2(CZ' + CN)^2 - 4e^2CZ'.CN \\ \text{i.e.} \quad S'N^2 + NP^2 - 4CS'.CN &= e^2Z'N^2 - 4e^2.CZ'.CN \\ \text{or} \quad S'P^2 - 4ae.CN &= e^2M'P^2 - 4e^2(\frac{a}{e})CN \\ \text{i.e.} \quad S'P^2 &= e^2M'P^2 \\ \frac{S'P^2}{M'P^2} &= e^2 = \frac{SP^2}{MP^2} \end{aligned}$$

If N lies on the same side of C as S' lies then a similar argument gives the same result. Thus

$$S'P = e.PM' \tag{8.6}$$

$$\begin{aligned} \text{Also} \quad SP + S'P &= e(PM + PM') \\ &= e.MM' = e.ZZ' \\ &= 2e.CZ = 2CA = 2a. \end{aligned}$$

So for every point P on the ellipse,

$$SP + S'P = 2a \tag{8.7}$$

Result (8.6) shows that we may regard S' as a focus and the line l' as a directrix. Thus an ellipse has two foci, each focus having its own directrix. Also the sum of the distances from the two foci is constant for every point on the ellipse.

The points A, A' are called the vertices of the ellipse and C is called the centre of the ellipse.

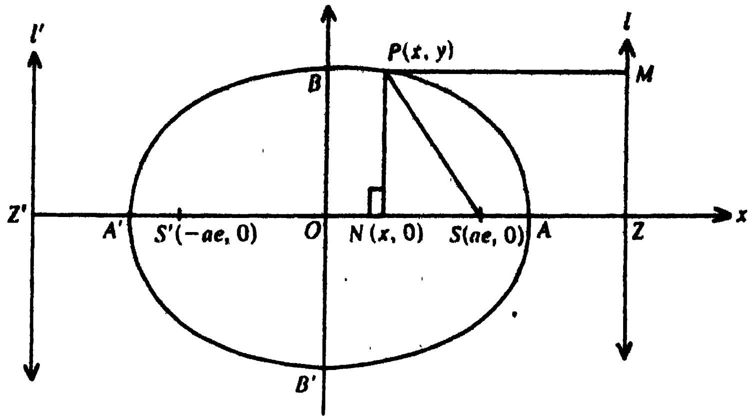


Fig. 8.8

To obtain the equation of the ellipse in its simplest form, we again choose our origin and the directions of the axes conveniently as follows.

Let S, S' be the foci, l and l' corresponding directrices, A, A' vertices and e the eccentricity $(0 < e < 1)$. Let O be the mid-point of AA' and suppose $AA' = 2a$. Then

$$\begin{aligned} OA &= OA' = a \\ OS &= OS' = ae \\ OZ &= OZ' = a/e \end{aligned}$$

We take O as the origin, direction OA as the positive direction of the x-axis.

Then coordinates of S, A and Z are respectively (ae, o), (a, o) and $(a/e, o)$.

We take any point $P(x, y)$ on the ellipse and draw perpendiculars PM on l, PN on the x-axis. Then

$$SP = e.PM$$

So $$SP^2 = e^2.PM^2$$

i.e. $$(ae - x)^2 + y^2 = e^2\left(\frac{a}{e} - x\right)^2$$

Therefore $$x^2(1 - e^2) + y^2 = a^2(1 - e^2)$$

or $$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1 \tag{8.8}$$

As $1 - e^2 > 0$, we can write $b = a\sqrt{1 - e^2}$. Then the above equation becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{8.9}$$

Thus every point $P(x, y)$ on the ellipse satisfies (8.9). Again if any point $P(x, y)$ satisfies (8.9), then it is easy to show (by retracing our steps) that $PS = ePM$, so that such a point P must be on the ellipse. So (8.9) is the equation of the ellipse in standard form.

Remarks

1. As $b = a\sqrt{1-e^2}$, so $0 < b < a$.

2. The ellipse intersects the x-axis at the vertices A and A'. The line AA' is called the major axis of the ellipse.

3. The ellipse intersects the y-axis where $x = 0$, i.e. where $y^2 = b^2$ or $y = \pm b$. These are points $B(0, b)$ and $B'(0, -b)$. The line BB' is called the minor axis of the ellipse.

4. For every point (x, y) on the ellipse

$$\frac{x^2}{a^2} = 1 - \frac{y^2}{b^2} \leq 1, \text{ i.e. } x^2 \leq a^2$$

   and so $-a \leq x \leq a$.
   Therefore, the ellipse lies between the lines $x = -a$ and $x = a$. Similarly, it lies between the lines $y = -b$ and $y = b$.

5. If (x, y) satisfies (8.8), then so does $(-x, y)$ and also $(x, -y)$. These show that the ellipse is symmetric about both the x and the y-axes.

6. We chose positive direction of the x-axis along OA. If we had chosen positive direction of the y-axis along OA, the equation of the ellipse would be

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

   The foci would be on the y-axis.

7. The eccentricity measures the flatness of the ellipse. Distance between the two foci SS' is $2ae$. As e increases, this distance increases, foci move away from the centre and the ellipse becomes flatter. Or, notice that $a^2 - b^2 = a^2 - a^2(1-e^2) = a^2e^2$. So as e increases, $a^2 - b^2$ increases, so the major axis becomes much longer compared to the minor axis and so the ellipse becomes more flat, more elongated. On the other hand, if $e \to 0$, the foci come closer to the centre, the difference between a and b tends to be zero, and the ellipse is almost like a circle. Indeed, we might consider circle as an ellipse with zero eccentricity.

Example 8.2

Given the ellipse with equation $9x^2 + 25y^2 = 225$, find the major and minor axes, eccentricity, foci and vertices.

Solution
We put the equation in standard form by dividing by 225 and get

$$\frac{x^2}{25}+\frac{y^2}{9}=1$$

which shows that $a=5$ and $b=3$. Hence, $9=25(1-e^2)$, so $e=\frac{4}{5}$. Since the denominator of x^2 is larger, the major axis is along x-axis, minor axis along y-axis, foci are $(4,0)$ and $(-4,0)$ and vertices $(5,0)$ and $(-5,0)$.

Example 8.3
Show that $4x^2+16y^2-24x-32y=12$ is the equation of an ellipse and find its vertices, foci, eccentricity and directrices.

Solution
The equation can be written as

$$4(x-3)^2+16(y-1)^2=64$$

or

$$\frac{(x-3)^2}{16}+\frac{(y-1)^2}{4}=1$$

Translating the coordinate axes so that the new coordinates are $X=x-3$, $Y=y-1$, the equation becomes $\frac{X^2}{16}+\frac{Y^2}{4}=1$ which is an ellipse with $a=4$, $b=2$, $4=16(1-e^2)$, so $e=\frac{\sqrt{3}}{2}$. In the new coordinates: foci are $(\pm 2\sqrt{3},0)$, vertices $(\pm 4,0)$ and directrices $X=\frac{8}{\sqrt{3}}$, $X=-\frac{8}{\sqrt{3}}$. Hence, in the original coordinate system (x,y), foci are $(3\pm 2\sqrt{3},1)$, vertices are $(3\pm 4,1)$, and directrices $x=3+\frac{8}{\sqrt{3}}$ and $x=3-\frac{8}{\sqrt{3}}$.

Example 8.4
Find the equation of the ellipse with foci at $(\pm 5,0)$ and $x=\frac{36}{5}$ as one directrix.

Solution
We have $ae=5$, $\frac{a}{e}=\frac{36}{5}$. $\therefore a^2=36$, so $a=6$
so, $e=\frac{5}{6}$.
Now $b=a\sqrt{1-e^2}=6\sqrt{1-\frac{25}{36}}=\sqrt{11}$
Thus, the equation is $\frac{x^2}{36}+\frac{y^2}{11}=1$

EXERCISE 8.2

1. For the following ellipses find the lengths of major and minor axes, coordinates of foci and vertices, and the eccentricity.

   (a) $16x^2 + 25y^2 = 400$

   (b) $9x^2 + 16y^2 = 144$

   (c) $3x^2 + 2y^2 = 6$

   (d) $x^2 + 4y^2 - 2x = 0$

2. Find the equation of the ellipse satisfying the given condition.

   (a) Vertices at $(\pm 5, 0)$, foci at $C(\pm 4, 0)$

   (b) Vertices at $(0, \pm 10)$, $e = \frac{4}{5}$

   (c) Foci at $(0, \pm 4)$, $e = \frac{4}{5}$

   (d) Axes along coordinate axes, passing through $(4, 3)$ and $(-1, 4)$

   (e) Foci at $(\pm 3, 0)$, passing through $(4, 1)$

   (f) $e = \frac{3}{4}$, foci on y-axis, centre at origin, passing through $(6, 4)$

3. Find the equation of the set of all points the sum of whose distances from $(3, 0)$ and $(9, 0)$ is 12.

4. Find the equation of the set of all points whose distances from $(0, 4)$ are $\frac{2}{3}$ of their distances from the line $y = 9$.

8.4 Hyperbola

Definition 8.3

A ***hyperbola*** is the set of all points in the plane whose distances from a fixed point in the plane bears a constant ratio, greater than one, to their distances from a fixed line in the plane.

The fixed point is called a focus, the fixed line a directrix and the constant ratio (denoted by e) the eccentricity of the hyperbola.

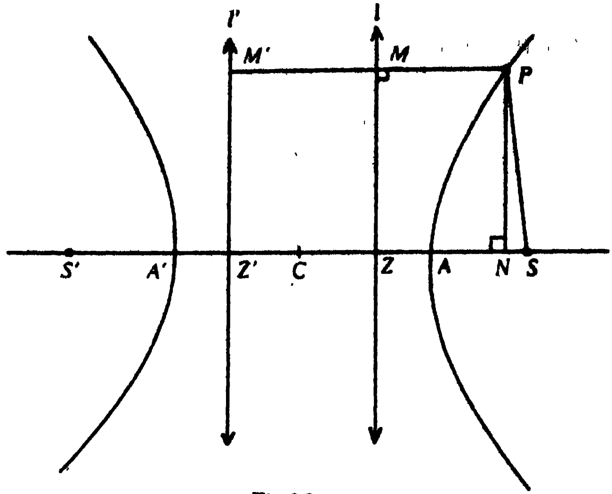


Fig 8.9

Before obtaining the equation of the hyperbola, we do some preparatory work. Let S be the focus and Z the foot of the perpendicular from S on the directrix l. Divide SZ internally and externally in the ratio $e : 1$, at the points A and A' respectively. As $e > 1$, A' will be beyond Z, i.e. Z will be between A' and S. Let $AA' = 2a$ and let C be the mid-point of AA .

From the manner of obtaining points A and A' they are points on the hyperbola (A, A' are called vertices of the hyperbola),

$$SA = e.AZ$$

$$SA' = e.A'Z$$

so $$SA + SA' = e(AZ + A'Z)$$

i.e. $$(CS - CA) + (CS + CA') = e.AA' = 2ae$$

or, as $$CA = CA', \text{ we get}$$

$$2CS = 2ae$$

or $$CS = ae$$

Also $$SA' - SA = e(A'Z - AZ)$$

i.e. $$AA' = e(CA' + CZ - CA + CZ)$$

or $$2a = 2e.CZ$$

or $$CZ = \frac{a}{e}$$

On the line CS, we take points S' and Z' so that they are symmetrically situated with respect to C to the points S and Z, i.e. such that C is the mid-point of SS' and also of ZZ'. Take line l' perpendicular to CS and through Z'.

Let P be any point on the hyperbola, PN perpendicular to CS, PM perpendicular to l and PM' perpendicular to l'.

As P is on the hyperbola,

$$SP^2 = e^2.PM^2$$

i.e.
$$PN^2 + NS^2 = e^2.NZ^2$$

If N lies on the same side of C as S lies, as shown in Fig. 8.9

$$PN^2 + (CS - CN)^2 = e^2(CN - CZ)^2$$

i.e.
$$PN^2 + (CS' - CN)^2 = e^2(CN - CZ')^2$$

i.e.
$$PN^2 + (CS' + CN)^2 - 4CS'.CN = e^2\{(CN + CZ')^2 - 4CN.CZ'\}$$

i.e.
$$PN^2 + S'N^2 - 4ae.CN = e^2.NZ'^2 - e^2.4.CN.\frac{a}{e}$$

i.e.
$$PS'^2 = e^2PM'^2$$

This shows that $\frac{PS'}{PM'} = e = \frac{PS}{PM}$

If N lies on the same side of C as S' lies then a similar argument gives the same result. Thus S' is another focus and l' the corresponding directrix.

Thus, like the ellipse, a hyperbola has two foci, each having its own directrix. Finally, note that

$$PS' - PS = e(PM' - PM) = e.MM' = e(ZZ')$$
$$= e.\frac{2a}{e} = 2a$$

so that the difference of the distances for every point on the hyperbola from the two foci is constant. A hyperbola can also be defined by this property. That is, a hyperbola is the set of all points in a plane the difference of whose distances from two fixed points in the plane is constant.

Now to obtain the equation of the hyperbola (with eccentricity $e > 1$) in its simplest form, we proceed as follows:

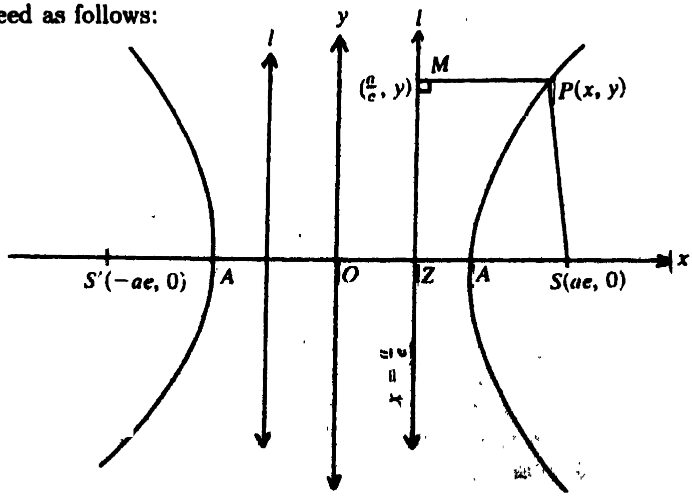


Fig. 8.10

Let S and S' be the foci of the hyperbola, A and A' vertices. Let O be the mid-point of SS' (called the centre of the hyperbola). We take O as the origin and OS the positive x-axis. Then S is (ae, o). Directrix l has the equation $x = \frac{a}{e}$. If $P(x, y)$ is any point on the hyperbola and PM is perpendicular on l, then M is $(\frac{a}{e}, y)$. The equation

$$SP^2 = e^2PM^2$$

becomes
$$(x - ae)^2 + y^2 = e^2(x - \frac{a}{e})^2 = (ex - a)^2$$

or
$$x^2(e^2 - 1) - y^2 = a^2(e^2 - 1)$$

or
$$\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2 - 1)} = 1$$

Note that as $e > 1$, $e^2 - 1 > 0$, so we can take $b = a\sqrt{e^2 - 1}$, then the above equation becomes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

which is the equation of the hyperbola.

Note:

1. As the equation shows, the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

is symmetrical about each axis.

2. As $\frac{x^2}{a^2} = 1 + \frac{y^2}{b^2} \geq 1$,

so for all points (x, y) on the hyperbola

$$|\frac{x}{a}| \geq 1,$$

i.e. $x \leq -a$ or $x \geq a$

The hyperbola, therefore, has two branches, one in the half plane $x \leq -a$ and the other in the half plane $x \geq a$.

3. As $b^2 = a^2(e^2 - 1)$, smaller the e, smaller the b, so

$$\frac{x^2}{a^2} = 1 + \frac{y^2}{b^2}$$

indicates that if e is small (but of course > 1), then $\frac{x^2}{a^2}$ is large so that for given y, x would be large. This means the point (x, y) on the hyperbola would be to the far right. Hence smaller eccentricity means the branches of the hyperbola would be bending towards the x-axis.

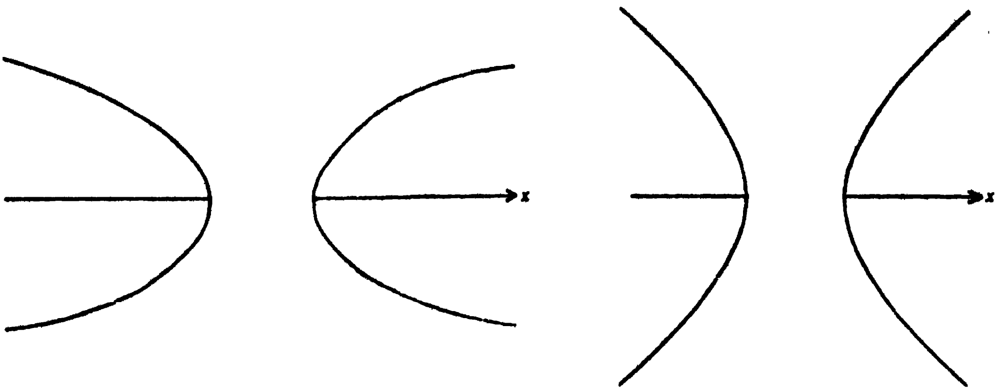


Fig 8.11

On the other hand, large e means small $\frac{y^2}{b^2}$ and so small $(\frac{x^2}{a^2} - 1)$, i.e. small x, so the branches would be opening up as the eccentricity increases. In the limit (as $e \to \infty$), the hyperbola would nearly coincide with the line $x = \pm a$.

4. We have taken the line of vertices as the x-axis; if we take it as the y-axis, the equation of the hyperbola would be of the form

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

with $b^2 = a^2(e^2 - 1)$.

5. Numbers 2a and 2b are called the length of transverse and the conjugate axes for the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Example 8.5

For the hyperbola $9x^2 - 16y^2 = 144$, find the vertices, foci and eccentricity

Solution

The equation can be written as

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

so $a = 4$, $b = 3$, $9 = 16(e^2 - 1)$, so that $e^2 = \frac{9}{16} + 1 = \frac{25}{16}$ giving e Vertices are $(\pm a, 0) = (\pm 4, 0)$ and foci are $(\pm ae, 0) = (\pm 5, 0)$.

Example 8.6

Find the equation of the hyperbola with vertices at $(0, \pm 6)$ and $e = \frac{5}{3}$. Find its foci.

Solution

Since the vertices are on the y-axis (with origin at the mid-point), the equation is of the form

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

As vertices are $(0, \pm 6)$ $a = 6$, $b^2 = a^2(e^2 - 1) = 36(\frac{25}{9} - 1) = 64$

so the equation is $\frac{y^2}{36} - \frac{x^2}{64} = 1$ and the foci are $(0, \pm ae) = (0, \pm 10)$.

Example 8.7

Find the equation of the hyperbola whose vertices are $(\pm 6, 0)$ and one of the directrices is $x = 4$.

Solution

As the vertices are on the x-axis and their mid-point is the origin, the equation is of the type

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

where $b^2 = a^2(e^2 - 1)$, vertices are $(\pm a, 0)$ and directrices are $x = \pm a/e$

$$\begin{aligned} \text{Thus} \qquad a &= 6, \frac{a}{e} = 4 \text{ and so } e = \frac{3}{2} \\ \text{Thus} \qquad b^2 &= 36(\frac{9}{4} - 1) = 45. \end{aligned}$$

Consequently, the equation is

$$\frac{x^2}{36} - \frac{y^2}{45} = 1.$$

EXERCISE 8.3

1. For the following hyperbolas find the lengths of transverse and conjugate axes, eccentricity and the coordinates of foci and vertices:

   (a) $16x^2 - 9y^2 = 144$

   (b) $2x^2 - 3y^2 - 6 = 0$

   (c) $3x^2 - 2y^2 = 1$

2. Find the equation of the hyperbola with

   (a) Vertices $(\pm 5, 0)$, foci $(\pm 7, 0)$

   (b) Vertices $(0, \pm 7), e = \frac{4}{3}$

   (c) Foci $(0, \pm 10)$, passing through (2,3)

3. Find the equation of the set of all points such that the difference of their distances from $(4,0)$ and $(-4,0)$ is always equal to 2.

4. Show that the equation $16x^2 - 3y^2 - 32x - 12y - 44 = 0$ represents a hyperbola. Find the lengths of axes and eccentricity.

8.5 Condition for Tangency of the Line $y = mx + c$

In what follows, we shall study the intersection of the line $y = mx + c$ with the given conic. Since the equation of every conic is of second degree, we shall in general find two points of intersection. Again, since a tangent is a line which intersects the curve in two coincident points, we shall use this property to find the condition for tangency of the line $y = mx + c$ to the given conic.

(a) *Parabola:* Consider the intersection of the line $y = mx + c$ with the parabola

$$y^2 = 4ax \tag{8.10}$$

Substituting the value of y from the equation of the line in the equation of the parabola, we find that x-coordinate of the points of intersection of the line with the parabola are the roots of the equation.

$$(mx + c)^2 = 4ax$$

$$\text{i.e. } m^2x^2 + 2(mc - 2a)x + c^2 = 0 \tag{8.11}$$

This quadratic equation generally gives two values of x, except when $m = 0$. When $m = 0$, the line is paralled to the axis of the parabola and hence it intersects the parabola in one point only. So, when $m \neq 0$, the line will be a tangent to the parabola if the roots of (8.11) are coincident. The condition for this is

$$(mc - 2a)^2 = m^2c^2$$

$$\text{i.e. } c = \frac{a}{m}$$

The line

$$y = mx + \frac{a}{m}$$

therefore is tangent to the parabola $y^2 = 4ax$ for all values of $m \neq 0$.

With this value of c, the two coincident roots of the quadratic equation (8.11) are

$$x = \frac{a}{m^2}.$$

Substituting this in the equation of the line, we get y :

Hence, the point of contact of the tangent $y = mx + \frac{a}{m}$ with the parabola $y^2 = 4ax$ is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

It may be noted in passing that the points $(at^2, 2at)$ for all values of t, lies on the parabola $y^2 = 4ax$. We say that the parametric equation of the parabola is

$$x = at^2,\ y = 2at$$

The point $(at^2, 2at)$ is usually referred to as the point "t" on the parabola.

(b) *Ellipse:* Consider the intersection of the line $y = mx + c$ with the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. As before, the x-coordinate of the point of intersection is given by

$$b^2x^2 + a^2(mx + c)^2 = a^2b^2$$

$$\text{or}\quad (a^2m^2 + b^2)x^2 + 2mxa^2c + a^2(c^2 - b^2) = 0$$

which gives two roots of x, and hence two points of intersection. In order that the line may be a tangent, i.e. the two points of intersection may be coincident, we must have

$$m^2c^2a^4 = a^2(c^2 - b^2)(a^2m^2 + b^2)$$

$$\text{or}\quad c^2 = a^2m^2 + b^2$$

Hence, the line $y = mx + \sqrt{a^2m^2 + b^2}$ is a tangent to the ellipse for all values of m.

As before, the point of contact of this tangent with the ellipse is

$$\left(\frac{-a^2m}{\sqrt{a^2m^2 + b^2}}, \frac{b^2}{\sqrt{a^2m^2 + b^2}}\right)$$

It may be noted that in fact we get two values of c with opposite signs, and hence the line $y = mx - \sqrt{a^2m^2 + b^2}$ is also a tangent to the ellipse. Thus we get two parallel tangents (as in the case of a circle).

It is seen that the point $(a\cos\theta, b\sin\theta)$ lies on the ellipse for all values of $\theta, 0 \le \theta < 2\pi$. Hence, the parametric equation of the ellipse is

$$x = a\cos\theta,\ y = b\sin\theta$$

where θ is called eccentric angle.

(c) *Hyperbola:* Considering the intersection of the line $y = mx + c$ with the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we find that the x-coordinate of the points of intersection is given by

$$b^2x^2 - a^2(mx + c)^2 - a^2b^2 = 0$$

$$\text{or}\quad (a^2m^2 - b^2)x^2 + 2mca^2x + a^2(b^2 + c^2) = 0$$

For the line to be a tangent, i.e. the two roots of this quadratic equation to be coincident, we must have

$$c^2 = a^2m^2 - b^2$$

Hence, the line $y = mx + \sqrt{a^2m^2 - b^2}$ is a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ for all values of m.

Proceeding as before, we get the point of contact

$$\left(\frac{-ma^2}{\sqrt{a^2m^2 - b^2}}, \frac{-b^2}{\sqrt{a^2m^2 - b^2}}\right)$$

Here also we shall get another parallel tangent

$$y = mx - \sqrt{a^2m^2 - b^2}$$

It may also be noted that the point $(a\sec\theta, b\tan\theta)$ lies on the hyperbola for all values of θ, and hence the parametric equation of the hyperbola can be written as

$$x = a\sec\theta, y = b\tan\theta$$

Example 8.8

Find the equation of the chord of the parabola $y^2 = 4ax$ joining the points $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$. Hence, deduce the equation to the tangent at $(at^2, 2at)$.

Solution

Required equation of the chord is

$$\begin{aligned} y - 2at_1 &= \frac{2at_2 - 2at_1}{at_2^2 - at_1^2}(x - at_1^2) \\ &= \frac{2}{t_1 + t_2}(x - at_1^2) \\ \text{or } (t_1 + t_2)y &= 2(x + at_1t_2) \end{aligned}$$

If $t_2 \to t_1$ this becomes $t_1y = x + at_1^2$.
Hence, the equation of tangent at $(at^2, 2at)$ is

$$ty = x + at^2$$

Example 8.9

Show that the line $x\cos\alpha + y\sin\alpha = p$ is tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if

$$p^2 = a^2\cos^2\alpha + b^2\sin^2\alpha$$

Solution

We can write the equation of the line as $y = mx + c$ where $m = -\cot\alpha$, $c = p\,\mathrm{cosec}\,\alpha$. It will be a tangent if

$$c^2 = a^2m^2 + b^2$$

i.e.
$$p^2\,\mathrm{cosec}^2\alpha = a^2\cot^2\alpha + b^2$$

or
$$p^2 = a^2\cos^2\alpha + b^2\sin^2\alpha$$

EXERCISE 8.4

1. Show that the equation of the tangent to the parabola $y^2 = 4ax$ which makes an angle θ with its axis is $y = x\tan\theta + a\cot\theta$.
2. Prove that the line $lx + my + n = 0$ will touch the parabola $y^2 = 4ax$ if $ln = am^2$.
3. Show that $y - x - 2 = 0$ is tangent to the parabola $y^2 = 8x$. What is the point of contact?
4. A line $lx + my + n = 0$ touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Show that $a^2l^2 + b^2m^2 = n^2$.
5. Find the equation of the tangent at the point $(at^2, 2at)$ to the parabola $y^2 = 4ax$.
6. Find the equation of the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at a point $(a\cos\theta, b\sin\theta)$.
7. Find the equation of the tangents to the ellipse $4x^2 + 3y^2 = 5$ which are parallel to the line $y = 3x + 7$. Also find the point of contact.
8. Find the equation of the tangent to the hyperbola $4x^2 - 9y^2 = 1$ which is parallel to the line $4y = 5x + 7$.

MISCELLANEOUS EXERCISES ON CHAPTERS 4, 5, 6, 7 and 8

1. $A(a, 0)$ and $B(-a, 0)$ are two given points. Show that the equation of the set of points P such that $PA + PB = c$ is $4(c^2 - 4a^2)x^2 + 4c^2y^2 = c^2(c^2 - 4a^2)$.
2. The slope of a line through $A(3, 2)$ is $\frac{3}{4}$. Find the coordinates of the points on the line that are 5 units away from A.
3. Find the equation of the set of points P where P is the mid-point of the portion of a line, distant p from the origin, intercepted between the axes.
4. Two equal sides of an isosceles triangle are given by the equations $7x - y + 3 = 0$, $x + y - 3 = 0$ and its third side passes through the point $(1, -10)$. Determine the equations of the third side.

5. Show that the area of the triangle formed by $ax^2+2hxy+by^2=0$ and $lx+my+n=0$ is given by
$$\frac{n^2\sqrt{h^2-ab}}{am^2-2hlm+bl^2}$$

6. Prove that two of the lines represented by the equation $ax^3+bx^2y+cxy^2+dy^3=0$ will be at right angles if $a^2+ac+bd+d^2=0$.

7. Find the equation to the circle which touch the axis of y at a distance $+4$ from the origin and cuts off an intercept 6 from the axis of x.

8. Find the equations to the tangents to the circle $x^2+y^2-6x+4y-12=0$ which are parallel to the line $4x+3y+5=0$.

9. Find the equation to the circle cutting orthogonally the three circles
$x^2+y^2-2x+3y-7=0$, $x^2+y^2+5x-5y+9=0$, and $x^2+y^2+7x-9y+29=0$

10. Find the equation of the parabola whose focus is $(5,2)$ and having the vertex at $(3,-2)$.

11. Show that the line $x+2y-4=0$ touches the ellipse $3x^2+4y^2=12$. Find the coordinates of the point of contact.

12. Prove that the product of the perpendiculars from the foci upon any tangent to the ellipse
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$
is b^2

13. Find the equation of the hyperbola whose directrix is $2x+y=1$, focus $(1,2)$ and eccentricity is $\sqrt{3}$.

14. Show that the line $x\cos\alpha+y\sin\alpha=p$ touches the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ if $p^2=a^2\cos^2\alpha-b^2\sin^2\alpha$.

15. The line $3x+4y=7$ is a normal to the hyperbola $4x^2-3y^2=1$ at a point P on it. Find the coordinates of P

CHAPTER 9

Complex Numbers

9.1 The Algebra of Complex Numbers

The concept of numbers, as you are aware, is gradually extended from natural numbers to integers, from integers to rational numbers and from rational numbers to real numbers. You might have observed that this gradual extension is based on mathematical needs. For instance, there is no solution to the equation $x^2 = 2$, if $x \in \mathbf{Q}$, the set of rational numbers. Thus we are forced to introduce irrational numbers and thereby get the set $\mathbf{R}$ of the real numbers which is the union of $\mathbf{Q}$ with the set of all irrational numbers. We know that the square of every real number is non-negative, hence there exists no real number with its square equal to -1. In other words, there is no solution in $\mathbf{R}$ of the equation $x^2 + 1 = 0$. Just as we introduced the irrational number $\sqrt{2}$ so as to be able to solve the equation $x^2 = 2$, in the same way we assume there is a number, which is not in $\mathbf{R}$, whose square is -1. This number is denoted by i. Thus $i^2 = -1$ and i is the square root of -1. For any two real numbers, a and b, we can form a new number $a + ib$. This number $a + ib$ is called a *complex number*. The set of all complex numbers is denoted by $\mathbf{C}$. If we write in the sequel $a+ib$ for an element of $\mathbf{C}$, it is tacitly assumed that a and b are real numbers unless otherwise stated. The real number a is called the *real part* of $a + ib$ and the real number b is called the *imaginary part*. We identify $a + i0$ with real number a and $i.1$ with i. A complex number is denoted by a single letter such as z, w and so on when it is convenient to do so.

The real part of a complex number z is denoted by $\operatorname{Re} z$ and the imaginary part as $\operatorname{Im} z$.

If $z = x + iy$, where x and y are real, then $\operatorname{Re} z = x$ and $\operatorname{Im} z = y$.

We define addition and multiplication between two complex numbers as follows:

$$\begin{aligned}(a+ib)+(c+id) &= (a+c)+i(b+d)\\ (a+ib)(c+id) &= (ac-bd)+i(ad+bc)\end{aligned}$$

It will be observed that while adding two complex numbers, the real and imaginary parts of the sum is obtained by adding separately the real and imaginary parts of the

ands. As regards multiplication, we have multiplied the two complex numbers as if they were real numbers and have used the fact that $i^2 = -1$. If we denote $i(-1)$ by $-i$ as is usual, $(-i)(-i) = (0+i(-1))\,(0+i(-1)) = 0-(-1)(-1)+i(0+0) = -1$, so that $-i$ is also a square root of -1. Now $a+c$, $b+d$, $ac-bd$, $ad+bc$ are obtained by using the binary operations of addition, multiplication, etc. in $\mathbf{R}$. Thus in $\mathbf{C}$, two binary operations $+$ and $\cdot$ have been defined. These two operations are named addition and multiplication in $\mathbf{C}$.

Let $z = a+ib$. If $b = 0$, then we say that z is a purely real number and is equal to a. If $a = 0$, then we say that z is a purely imaginary number and is equal to ib. We, more often, prefer to write bi such as $2i, 3i$, etc. If both $a = 0$ and $b = 0$, then instead of writing $0+i0$, we simply write 0. Two complex numbers $a+ib$ and $c+id$ are equal if and only if $a = c$ and $b = d$. In particular, $a+ib = 0$ if and only if $a = 0$ and $b = 0$.

We now consider the operation of addition in $\mathbf{C}$. For any three complex numbers $a+ib$, $c+id$, $e+if$,

$$\begin{aligned}
&\{(a+ib)+(c+id)\}+(e+if)\\
&= \{(a+c)+i(b+d)\}+(e+if)\\
&= (a+c+e)+i(b+d+f)\\
&= (a+ib)+(c+e)+i(d+f)\\
&= (a+ib)+\{(c+id)+(e+if)\}
\end{aligned}$$

so that the operation is associative. It is also commutative, as is easily seen. We also see that

$$\begin{aligned}
(a+ib)+0 &= (a+ib)+(0+i0) = a+0+i(b+0)\\
&= a+ib\\
&= 0+(a+ib)
\end{aligned}$$

Thus 0 is an identity element for addition in $\mathbf{C}$. If $z \in \mathbf{C}$, $z = a+ib$, we write $-z = -a+i(-b) = -a-ib$ and observe that

$$\begin{aligned}
z+(-z) = z-z &= (a+ib)+\{-a+i(-b)\}\\
&= 0+i0 = -z+z.
\end{aligned}$$

It is a good exercise to check that 0 is the only identity element for addition and that if $z+z' = 0$, then $z' = -z$. That $z+(-z) = -z+z = 0$ is expressed alternatively by saying that $-z$ is the additive inverse of z. Thus the binary operation of addition in $\mathbf{C}$ is associative, commutative, possesses identity element and is such that every element of $\mathbf{C}$ has additive inverse.

Let us now turn to multiplication. If $a+ib$, $c+id$, $e+if$ are any three complex numbers,

$$\begin{aligned}
(a+ib)\{(c+id)(e+if)\} &= (a+ib)\{(ce-df)+i(cf+de)\}\\
&= a(ce-df)-b(cf+de)+i\{a(cf+de)+b(ce-df)\}\\
&= ace-adf-bcf-bde+i(acf+ade+bce-bdf).
\end{aligned}$$

Likewise

$$\begin{aligned}\{(a+ib)(c+id)\}(e+if) &= (ac-bd)e-(ad+bc)f+i\{(ac-bd)f+(ad+bc)e\}\\ &= (ace-bde-adf-bcf)+i(acf-bdf+ade+bce)\end{aligned}$$

Thus,

$$\{(a+ib)(c+id)\}(e+if) = (a+ib)\{(c+id)(e+if)\}$$

Thus multiplication in $\mathbf{C}$ is associative. It can also be shown that it is commutative. As said earlier, denoting $1+i0$ by 1, we see that for any complex number $a+ib$,

$$1.(a+ib) = (a+ib).1 = (a+ib)(1+i0) = (a.1-b.0)+i(a.0+b.1) = (a+ib).$$

Similarly, it can be shown that $0.(a+ib) = (a+ib).0 = 0+i0 = 0$ for any complex number $a+ib$.

In other words, 1 is an identity element for the binary operation of multiplication in $\mathbf{C}$ and it is, in fact, the only one. For, if $a+ib \neq 0$ (this assumption may be made since $0.z = z.0 = 0$ for any complex number z)

and $\qquad (a+ib)(c+id) = (a+ib),$

then $\qquad ac-bd = a,\ ad+bc = b$

i.e. $\qquad a(c-1) = bd, \qquad (A)$

$\qquad b(c-1) = -ad. \qquad (B)$

Multiplying both sides of (A) by a and of (B) by b and adding, we get

$$(a^2+b^2)(c-1) = 0.$$

Since $a+ib \neq 0$, $a^2+b^2 \neq 0$, so $c = 1$. It turns out that $ad = 0$ and since a is not always 0; $d = 0$. So $c+id = 1$. Thus we see that the number 1 is unique. In this connection note that if k is any real number $k(a+ib) = ka+ikb$. In other words, multiplying a complex number z by a real number k amounts to multiplying the real and imaginary parts of z by k.

If $z = a+ib$ is a complex number, we denote $a+i(-b) = a-ib$ by $\bar{z}$ and call it the conjugate of z. It may be noted that the conjugate of $\bar{z}$ is z, i.e. $\overline{(\bar{z})} = z$. If z is real, i.e. if $b = 0$, then evidently $z = \bar{z}$. Conversely, if $z = \bar{z}$, i.e. if $a+ib = a-ib$, then $b = -b$ so that $b = 0$.

To sum up, a complex number z is its own conjugate if and only if it is real. Since $z+\bar{z} = (x+iy)+(x-iy) = 2x$,

$$\operatorname{Re} z = x = \frac{z+\bar{z}}{2}$$

Similarly, $\quad \operatorname{Im} z = y = \dfrac{z-\bar{z}}{2i}$

Also, if $z = a+ib$.

$$z\bar{z} = (a+ib)(a-ib) = a^2+b^2 = \bar{z}\,z,$$

since multiplication is commutative. Hence, we conclude that for any complex number z, the product $z\bar{z}$ is always a non-negative real number.
If $z \in \mathbf{C}, z = a + ib, a, b$ real, it is customary to denote the non-negative square root of $a^2 + b^2$ by $|z|$, and call it the *modulus* or absolute value of the complex number z.
Thus,

$$z\bar{z} = |z^2| \quad \text{and} \quad |z| = \sqrt{a^2 + b^2}$$

for any $z \in \mathbf{C}$. If $z \neq 0$, at least one of the two real numbers a and b is non-zero, so that $|z| \neq 0$. (In fact, $|z| = 0$ if and only if $z = 0$, as is easily verified). Thus if $z \neq 0$

$$z.\frac{\bar{z}}{|z|^2} = 1 = \frac{\bar{z}}{|z|^2}z.$$

If $z = a + ib \neq 0$, the complex number $\frac{\bar{z}}{|z|^2} = \frac{a}{a^2 + b^2} - i\frac{b}{a^2 + b^2}$
is again a complex number which is a multiplicative inverse of z, in the sense explained earlier. The inverse of z is denoted by z^{-1} or $\frac{1}{z}$. Thus $zz^{-1} = 1$. The uniqueness of the multiplicative inverse of $z \neq 0$ is easily verified. We know that a complex number $a + ib = 0$ if and only if $a = 0$ and also $b = 0$. This is equivalent to saying that $a + ib = 0$ if and only if $a^2 + b^2 = 0$. Hence, $a + ib \neq 0$ if and only if $a^2 + b^2 \neq 0$. In other words, $|z| = 0$ if and only if $z = 0$.

We are now in a position to define the binary operation of *division* in the set of complex numbers. As in the case of real numbers, this definition enables a complex number to be divided by a non-zero complex number. We define $\frac{z}{w}$ as $z.w^{-1}$. Thus if $z = x + iy, w = u + iv, u + iv \neq 0$, i.e. $u^2 + v^2 \neq 0$, then

$$\begin{aligned}\frac{z}{w} = z.w^{-1} = (x+iy).\frac{\bar{w}}{|w|^2} &= (x+iy)\left(\frac{u}{u^2+v^2} - i\frac{v}{u^2+v^2}\right)\\ &= \left(\frac{xu+yv}{u^2+v^2}\right) + i\left(\frac{yu-xv}{u^2+v^2}\right)\end{aligned}$$

In particular, $\frac{1}{w} = \frac{u}{u^2+v^2} - i\frac{v}{u^2+v^2}$ as seen earlier.

We can now prove that the following simple and important result. If z and w are two complex numbers such that their product $zw = 0$, then at least one of the numbers z, w is zero. Suppose, as we may, that $z \neq 0$. Then z^{-1} exists. Now $zw = 0$. So $z^{-1}(z.w) = 0$. By associativity of multiplication $(z^{-1}z)w = 0$ or $1.w = 0$, i.e. $w = 0$. Thus if $z \neq 0$, then w must be equal to zero. As a corollary of this result we can say that the product of two non-zero complex numbers is not zero. That is, if $z \neq 0$, $w \neq 0$, then $zw \neq 0$.

Besides these properties, we have the following which connect addition and multiplication:

$$\text{if} \quad z_1, z_2, z_3 \in \mathbf{C},$$

then ,
$$z_1(z_2+z_3) = z_1z_2+z_1z_3,$$
$$(z_1+z_2)z_3 = z_1z_3+z_2z_3$$

These properties are known as *distributive properties* and signify that multiplication distributes over addition. These properties are verified easily from the definition of addition and multiplication in **C**. Addition and multiplication are related by the distributive laws, making **C** what is familiarly known as a *field*. Note that **R** too is a field. Every complex number $a+i0$ can be identified as stated earlier with the real number a. It can also be seen that $(a+i0)+(b+i0)=a+b+i0$ and $(a+i0)(b+i0)=ab+i0$. Hence, the sum and product of complex numbers $a+i0$ and $b+i0$ can be identified with the sum and product of the real numbers a and b. Hence **R**, the set of real numbers, can be considered to be a subset of **C**, the set of complex numbers.

You may recall that in defining addition and multiplication of complex numbers, we have used the fact that $i^2=-1$. Also note that $(-i)^2=-1$.

For two complex numbers z_1 and z_2, we have

$$\overline{(z_1+z_2)} = \bar{z}_1+\bar{z}_2\ ;\ \overline{(z_1-z_2)} = \bar{z}_1-\bar{z}_2$$

and

$$\overline{z_1z_2} = \bar{z}_1\bar{z}_2\ ;\ \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}\ \text{where } z_2 \neq 0.$$

These results can be verified by setting $z_1 = x_1+iy_1$ and $z_2 = x_2+iy_2$.

Let us recall that for a complex number z, $z\bar{z} = |z^2|$. Now, for two complex numbers z_1 and z_2, we have

$$\begin{aligned}|z_1z_2|^2 &= (z_1z_2)\overline{(z_1z_2)} = z_1z_2\bar{z}_1\bar{z}_2\\ &= z_1\bar{z}_1.z_2\bar{z}_2 = |z_1|^2.|z_2|^2\end{aligned}$$

Taking square root and noting that positive sign can be taken on both sides, we get $|z_1z_2| = |z_1|.|z_2|$

The above result is expressed in words as follows:

The absolute value of a product of two complex numbers is equal to the product of the absolute values of the numbers.

If $z_2 \neq 0$, then $z_1 = \left(\frac{z_1}{z_2}\right)z_2$ and so

$$|z_1| = \left|\frac{z_1}{z_2}\right|.|z_2|.$$

Hence,

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

Before we give a formula for the absolute value of the sum of two complex numbers, let us recall that the real and imaginary parts of a complex number $z = x + iy$ are given by $\text{Re}\, z = \frac{z+\bar{z}}{2}$ and $\text{Im}\, z = \frac{z-\bar{z}}{2}$. Hence, $2\,\text{Re}\, z = z + \bar{z}$, and $2i\,\text{Im}\, z = z - \bar{z}$. Now, for two complex numbers z_1 and z_2

$$\begin{aligned} |z_1 + z_2|^2 = (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) &= z_1\bar{z}_1 + z_2\bar{z}_2 + z_1\bar{z}_2 + \bar{z}_1 z_2 \\ &= |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \overline{(z_1\bar{z}_2)} \\ &= |z_1|^2 + |z_2|^2 + 2\,\text{Re}\, z_1\bar{z}_2 \end{aligned} \tag{9.1}$$

We have also

$$\begin{aligned} |z_1 - z_2|^2 = (z_1 - z_2)(\bar{z}_1 - \bar{z}_2) &= z_1\bar{z}_1 + z_2\bar{z}_2 - (z_1\bar{z}_2 + \bar{z}_1 z_2) \\ &= |z_1|^2 + |z_2|^2 - 2\,\text{Re}\, z_1\bar{z}_2 \end{aligned} \tag{9.2}$$

Adding (9.1) and (9.2) we get

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left(|z_1|^2 + |z_2|^2\right) \tag{9.3}$$

Again $x^2 \le x^2 + y^2$ and $y^2 \le x^2 + y^2$ and taking positive square roots, we get $|x| \le |z|$ and $|y| \le |z|$ leading to the conclusion that $-|z| \le |x| \le |z|$ and $-|z| \le y \le |z|$. The $\text{Re}\, z = |z|$ implies $x^2 = x^2 + y^2$ if $z = x + iy$, so that $y = 0$.
Using the inequality $-|z| \le \text{Re}\, z \le |z|$ in (9.1), we get

$$|z_1 + z_2|^2 \le |z_1|^2 + |z_2|^2 + 2|z_1||z_2| = (|z_1| + |z_2|)^2,$$
$$(\text{since} \quad |\bar{z}_2| = |z_2|)$$

Taking positive square roots, we obtain

$$|z_1 + z_2| \le |z_1| + |z_2| \tag{9.4}$$

(9.4) is called *triangle inequality*. The reasons for this nomenclature will be explained later.

Of the two complex numbers z_1 and z_2,

$$z_1 = z_1 - z_2 + z_2, \quad \text{i.e.} \quad |z_1| \le |z_1 - z_2| + |z_2|$$

Thus, $|z_1| - |z_2| \le |z_1 - z_2|$.
By interchanging z_1 and z_2, we get

$$|z_2| - |z_1| \le |z_2 - z_1|.$$

Combining the two inequalities, we get

$$||z_1| - |z_2|| \le |z_1 - z_2| \tag{9.5}$$

for any two complex numbers z_1 and z_2. The inequality (9.5) is also called triangle inequality. The inequality (9.4) can be extended to n complex numbers by finite induction. In other words,

$$|z_1 + z_2 + \ldots + z_n| \le |z_1| + |z_2| + \ldots + |z_n| \tag{9.6}$$

for any n complex numbers $z_1, z_2, \ldots z_n$.

Example 9.1

Express (i) $(-1+2i)+\left(\frac{1}{2}-i\right)$

(ii) $\left(\frac{1}{2}+\frac{1}{4}i\right)\left(-\frac{2}{3}-\frac{1}{4}i\right)$

in standard form.

Solution

(i) $(-1+2i)+(\frac{1}{2}-i)$

$$\begin{aligned} &= (-1+\frac{1}{2})+(2-1)i \qquad \text{(by definition)}\\ &= -\frac{1}{2}+1i=-\frac{1}{2}+i \end{aligned}$$

(ii)

$$\begin{aligned} &\left(\frac{1}{2}+\frac{1}{4}i\right)\left(-\frac{2}{3}-\frac{1}{4}i\right)\\ &= \left(\frac{1}{2}\cdot-\frac{2}{3}-\frac{1}{4}\cdot-\frac{1}{4}\right)+\left(\frac{1}{2}\cdot-\frac{1}{4}+\frac{1}{4}\cdot-\frac{2}{3}\right)i \qquad \text{(by definition)}\\ &= \left(-\frac{1}{3}+\frac{1}{16}\right)+\left(-\frac{1}{8}-\frac{1}{6}\right)i\\ &= -\frac{13}{48}-\frac{7}{24}i \end{aligned}$$

Remark

When a, b are specific real numbers, we prefer to write $a+bi$ to $a+ib$, e.g. we write $2+3i$ instead of $2+i3$. Further, we prefer to write $1.i$ as i.

Example 9.2

Show that for any real θ, $(\cos\theta+i\sin\theta)(\cos\theta-i\sin\theta)=1$

Solution

$$\begin{aligned} &(\cos\theta+i\sin\theta)(\cos\theta-i\sin\theta)\\ &= (\cos^2\theta+\sin^2\theta)+i(-\cos\theta\sin\theta+\sin\theta\cos\theta)\\ &= 1+0.i=1, \quad \text{since} \quad \cos^2\theta+\sin^2\theta=1 \quad \text{for any} \quad \theta. \end{aligned}$$

Remark

This example shows that if $z = \cos\theta + i\sin\theta$, then $\frac{1}{z} = \cos\theta - i\sin\theta$. If we denote $\cos\theta + i\sin\theta$ by $e^{i\theta}$, this means $\frac{1}{e^{i\theta}} = e^{i(-\theta)}$ which we may write as $e^{-i\theta}$

Example 9.3

Show that $\overline{(2+3i)(1-i)} = (2-3i)(1+i)$

Solution

$\overline{(2+3i)(1-i)} = \overline{(2+3)+(-2+3)i} = \overline{5+i} = 5-i$

Again, $(2-3i).(1+i) = (2+3)+(2-3)i = 5-i$, proving the equality.

Remark

The example suggests that if a complex number is given in terms of sums, products or quotients of complex numbers in standard form (i.e. of the form $a+ib$ where a, b are real), then its conjugate is obtained by changing i to $-i$ in all the constituents.

Example 9.4

Find the reciprocal or inverse of $1-i$.

Solution

$$\text{Reciprocal or inverse of } 1-i = \frac{1}{1-i} = \frac{1+i}{(1-i)(1+i)} = \frac{1+i}{2} = \frac{1}{2} + \frac{1}{2}i$$

EXERCISE 9.1

1. If $z_1, z_2 \in \mathbf{C}$, show that $(z_1+z_2)^2 = z_1^2 + 2z_1z_2 + z_2^2$
2. Show that $(1-i)^2 = -2i$
3. Find the conjugate of $\frac{1-i}{1+i}$
4. Show that if $a, b, c, d \in \mathbf{R}$, $\overline{(a+ib)(c+id)} = (a-ib)(c-id)$.

9.2 The Argand Diagram and the Polar Form

One can think of another notation for a complex number. Instead of writing $a+ib$, a, b real, we could write an ordered pair (a, b) of real numbers as standing for a complex number. We can rewrite the definition of addition and multiplication as follows:

$$\begin{aligned}(a,b)+(c,d) &= (a+c, b+d)\\ (a,b).(c,d) &= (ac-bd, ad+bc)\end{aligned}$$

and develop whatever was done in Section 9.1 in this notation. This notation is suggestive of the fact that a complex number can be represented by a point (a, b) in a plane referred to a pair of rectangular axes. If a complex number $z = a+ib$, then the point P with coordinates (a, b) is taken to represent the complex number $a+ib$. Since $OP = \sqrt{a^2+b^2}$, we have $|z| = OP$. This representation of complex numbers as points in the plane is known as the *Argand diagram*. The plane is called the *complex plane*. Note that only one complex number corresponds to a given point in the Argand diagram and vice versa. Thus we have a bijection between $\mathbf{C}$ and $\mathbf{R}^2$ $(= \mathbf{R} \times \mathbf{R}.)$

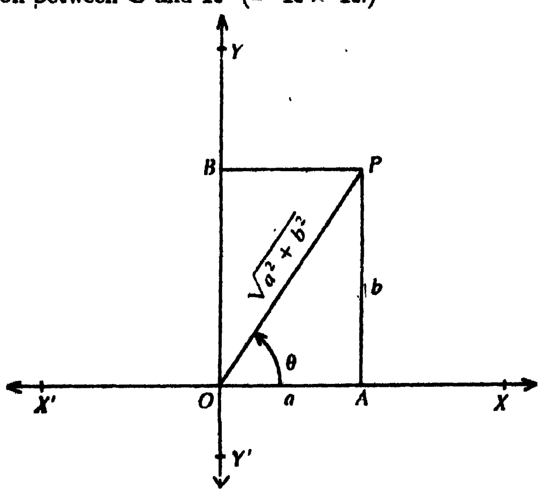


Fig. 9.1

Note that if $z \in \mathbf{C}$ is represented by P in the Argand diagram, $\bar{z}$, the conjugate of z is represented by $\bar{P}$ which is symmetric to P with respect to the axis $x'Ox$ or $\bar{P}$ is the (mirror) reflection of P on the axis $x'Ox$. (See Fig. 9.2).

It is customary to call the axis $x'Ox$ the *real axis* and $y'Oy$ the *imaginary axis* for the reason that points on $x'Ox$ corresponds to real numbers while those on $y'Oy$ to purely imaginary numbers.

Let P_1, P_2 represents two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ respectively in the Argand diagram in the complex plane as shown in the Fig. 9.3. Join the origin O with the points P_1 and P_2, respectively, and complete the parallelogram by drawing a line parallel to OP_1 from P_2 and another parallel to OP_2 from P_1, which intersect at the point P. It is clear from Fig. 9.3 that the coordinates of P are $(x_1 + x_2, y_1 + y_2)$.
Hence, the point P represents the complex number $(x_1 + x_2) + i(y_1 + y_2) = z_1 + z_2$.

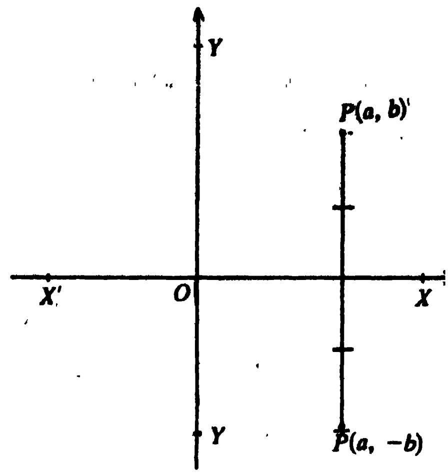


Fig 9.2

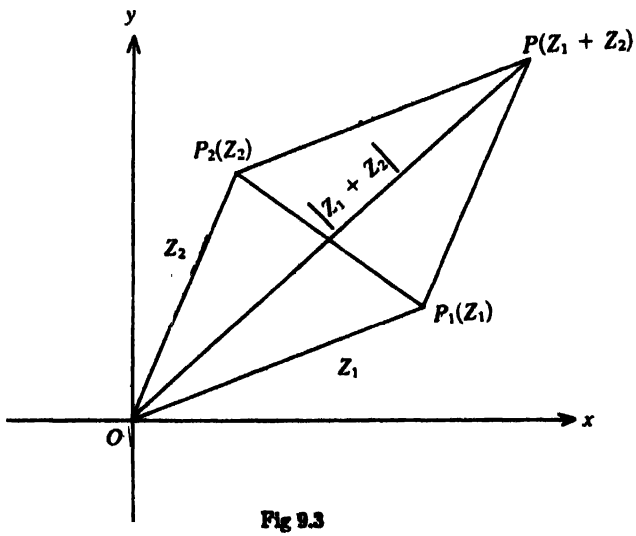


Fig 9.3

The length of the diagonal OP of the parallelogram OP_1PP_2 or the distance of P from O is equal to $|z_1 + z_2|$. Now the segment OP_2 = the segment P_1P. Hence, the length of each of them is equal to $|z_2|$. We know from school geometry that the sum of two sides of a triangle is greater than the third side. Hence, in ΔOPP_1, $OP \leq OP_1 + P_1P$.

This is the inequality $|z_1 + z_2| \leq |z_1| + |z_2|$ (See inequality (9.4) of Sec. 9.1). This is the reason why this inequality is called the triangle inequality.

In Fig. 9.4 P_2' represents $-z_2$. Completing the parallelogram $OP_2'QP_1$, we find that the point Q represents the sum of the complex numbers z_1 and $-z_2$ or $z_1 - z_2$. The length of segment OQ is equal to $|z_1 - z_2|$. Since OQ = the segment P_1P_2, it follows that the

length of the segment P_1P_2 is equal to $|z_1 - z_2|$. A theorem in geometry says that the sum of the squares on the diagonal of a parallelogram is equal to the sum of the squares on the four sides.

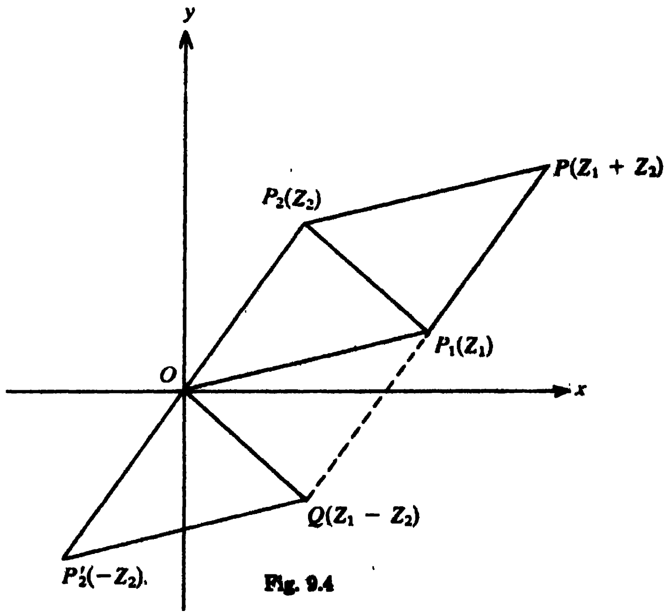


Fig. 9.4

Now in the parallelogram OP_1PP_2 of Fig. 9.3, the length of the diagonal OP is equal to $|z_1 + z_2|$ and we have just seen that the length of the other diagonal joining the points P_1 with P_2 is equal to $|z_1 - z_2|$. Hence, the above theorem is expressed by the equality

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$$

This equality was proved earlier (See equality (9.3) of Section 9.1). Hence, the geometrical interpretation of this equality is the theorem regarding the squares of diagonals of a parallelogram.

Another theorem of geometry states that the absolute difference of two sides of a triangle is less than the third side. In the triangle formed by the points O, P_1 and P_2 in Fig. 9.3 we find that the length of the side OP_1 is equal to $|z_1|$ and that of the side OP_2 is $|z_2|$. Also the length of the side P_2P_1 is $|z_1 - z_2|$. Hence, using the theorem mentioned above, we get

$$||z_1| - |z_2|| \leq |z_1 - z_2|$$

This is the inequality (9.5) of Section 9.1. Conversely, it should be now clear that the geometrical interpretation of the above inequality is the theorem stating that the absolute difference of two sides of a triangle is less than the third side. It is also called *triangle inequality*.

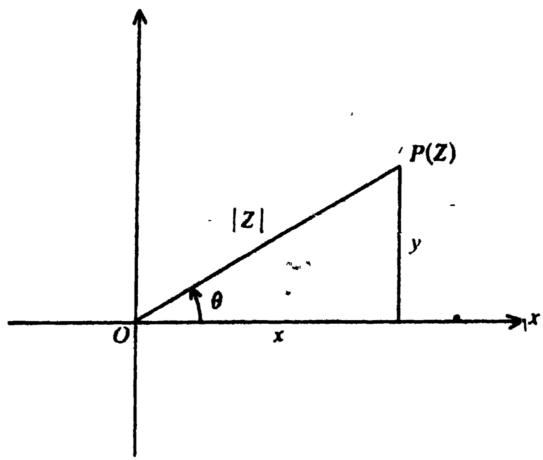


Fig. 9.5

9.3 Polar Representation

The introduction of polar coordinates to represent complex numbers in the plane is helpful in various ways. First, it is helpful in giving a geometrical interpretation of the product of two complex numbers. Second, the powers and roots of complex numbers can be very conveniently obtained with the help of their polar representations. Let P represent in the complex plane a non-zero complex number $z = x + iy$ and let the directed line OP make with the positive direction of the x-axis an angle θ. θ is measured in radians. Note that 2π radians are 360°. If we denote the length of the segment OP by r, then $r = \sqrt{x^2 + y^2} = |z|$ and $x = r\cos\theta$, $y = r\sin\theta$ and $\tan\theta = \dfrac{y}{x}$. (r, θ) are called the polar coordinates of the complex number z. $r(\cos\theta + \sin\theta)$ is the polar form or polar representation of z.

Clearly, $r > 0$, and as θ, the angle POx is measured in radians, $0 \leq \theta < 2\pi$. For every positive value of r and each value of θ between 0 and 2π, we get a unique point in the complex plane with polar coordinates as (r, θ) and conversely, every point in the plane has polar coordinates (r, θ), where $r \geq 0$ and $0 \leq \theta < 2\pi$. The segment OP is called the radius vector of the point P. The number $r = 0$ and if only if $z = 0$. This is in keeping with r being the modulus of z and θ is called its *argument* or *amplitude*. We write $\arg z = \theta$.

Example 9.5

Represent the complex number 1 + *i* in the polar form.

Solution

Let (r, θ) be the required coordinates. Then $r \cos \theta = 1$ and $r \sin \theta = 1$. Hence, $r^2(\cos^2\theta + \sin^2\theta) = 2$, or $r = \sqrt{2}$ (r is always ≥ 0) and $\cos \theta = \dfrac{1}{\sqrt{2}}$, $\sin \theta = \dfrac{1}{\sqrt{2}}$ and $\tan\theta = 1$. Hence, $\theta = \tan^{-1} 1 = \dfrac{\pi}{4}$. Thus polar coordinates of $1 + i$ are $(\sqrt{2}, \pi)$ and its polar form is $\sqrt{2} \left(\cos \dfrac{\pi}{4} + i \sin \dfrac{\pi}{4}\right.$

Remark

In the above example the argument of the complex number $1 + i$ is $\theta = \dfrac{\pi}{4}$ You can also see that $\dfrac{\pi}{4} + 2n\pi$, $n = 0, \pm 1, \pm 2$, is also an argument of $1 + i$. That value of θ which is such that $-\pi < 0 \leq \pi$ is called the *principal value* of arg z and is written sometimes as Arg z. The principal value of arg z in the above example is $\dfrac{\pi}{4}$.

Let us consider the polar form of the product of two complex numbers z_1 and z_2 before we give a geometrical construction of their product $z_1 z_2$. Let

$z_1 = r_1 (\cos \theta_1 + 1 \sin \theta_1)$ and $z_2 = r_2 (\cos \theta_2 + 1 \sin \theta_2)$.

Then

$$\begin{aligned} z_1 z_2 &= r_1 r_2 (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2) \\ &= r_1 r_2 (\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i(\sin \theta_1 \cos \theta_2 + \sin \theta_2 \cos \theta_1) \end{aligned}$$

From formulae for $\cos(\theta_1 + \theta_2)$ and $\sin(\theta_1 + \theta_2)$ we have

$$z_1 z_2 = r_1 r_2 \{\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)\} \tag{9.8}$$

Hence,

$$| z_1 z_2 | = r_1 r_2$$

and

$$\arg (z_1 z_2) = \theta_1 + \theta_2 = \arg z_1 + \arg z_2$$

If we write $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$, (9.8) shows that $z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$.

Here the relation (9.8) is to be interpreted in the following manner:

Any argument of $z_1 z_2$ can be expressed as any argument of z_1 plus any argument of z_2 and conversely, any argument of z_1 plus any argument of z_2 is equal to an argument of $z_1 z_2$.

It may be remarked that equation (9.8) may not be valid if we use 'Arg' instead of 'arg', i.e. if we take principal values.

Now we give a geometrical representation of the product $z_1 z_2$. We will use relations (9.7) and (9.8) for this purpose. Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$.

Let the points P_1 and P_2 represent the complex numbers z_1 and z_2. In Fig. 9.6, O is the origin and I is the point representing the number 1 on the real axis. Then the points O, P_1, I are the vertices of a triangle, viz. ΔOP_1I. Denote by ψ the angle OIP_1 as shown in Fig. 9.6.

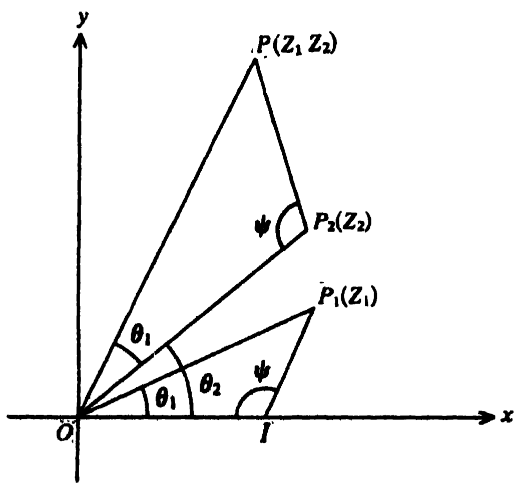


Fig. 9.6

Draw a line OP from O making angle equal to θ_1 with the segment OP_2, and another line from the point P_2 making an angle equal to ψ with the segment P_2O, intersecting the line OP at the point P. Then the point P represents the complex number $z_1 z_2$.

To prove this we note that ΔOP_1I is similar to ΔOPP_2, since $\angle P_1OI = \angle POP_2 = \theta_1$ and $\angle P_1IO = \angle PP_2O = \psi$. Thus the three angles of one triangle are respectively equal to the three angles of the other triangle. Hence,

$$\frac{OP}{r_1} = \frac{r_2}{1}$$

Thus

$$OP = r_1 r_2 \quad \text{and} \quad \angle POI = \theta_1 + \theta_2$$

Hence, P represents the complex number $r_1 r_2\{\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)\}$. It follows from relations (9.7) and (9.8) that P represents the complex number $z_1 z_2$.

Before giving a construction for $\dfrac{z_1}{z_2}$, $(z_2 \neq 0)$, we observe that for $z \neq 0$,

$$\frac{1}{z} = \frac{1}{r(\cos\theta + i\sin\theta)}$$

i.e.

$$\frac{1}{z} = \frac{\cos\theta - i\sin\theta}{r(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)}$$

i.e.

$$\frac{1}{z} = \frac{\cos\theta - i\sin\theta}{r(\cos^2\theta + \sin^2\theta)}$$

So

$$\frac{1}{z} = \frac{1}{r}\{\cos(-\theta) + i\sin(-\theta)\}$$

since

$$\cos(-\theta) = \cos\theta, \quad \sin(-\theta) = -\sin\theta$$

Hence, for $z_2 \neq 0$, we have

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{r_1}{r_2}(\cos\theta_1 + i\sin\theta_1)\{\cos(-\theta_2) + i\sin(-\theta_2)\}\\ &= \frac{r_1}{r_2}\{\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\}\end{aligned}$$

The corresponding geometrical construction is given in Fig. 9.7.
In Fig. 9.7, the point P represents the complex number $\dfrac{z_1}{z_2}$. In the two similar triangles OPP_1 and OIP_2 in which $OI = 1$, we have

$$\frac{OP}{OI} = \frac{OP_1}{OP_2} = \frac{|z_1|}{|z_2|} = \frac{r_1}{r_2}.$$

and the line OP makes with the real axis an angle equal to $\theta_1 - \theta_2$

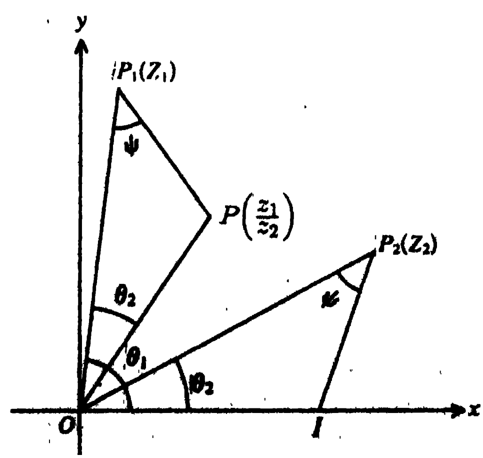


Fig 9.7

9.4 Powers and Roots of Complex Numbers

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then it follows from relation 9.7 of Section 9.3 that

$$\begin{aligned}
z_1z_2 &= r_1r_2\{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\} \\
\text{if} \quad z_3 &= r_3(\cos\theta_3 + i\sin\theta_3) \quad \text{is another complex number, then it follows that} \\
z_1z_2z_3 &= r_1r_2\{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\}\, r_3(\cos\theta_3 + i\sin\theta_3) \\
&= r_1r_2r_3\{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\}(\cos\theta_3 + i\sin\theta_3) \\
&= r_1r_2r_3\{\cos(\theta_1+\theta_2+\theta_3) + i\sin(\theta_1+\theta_2+\theta_3)\}
\end{aligned}$$

Using induction, it can be shown that the above formula can be extended to an arbitrary product of finite number of complex numbers. That is, if $z_1, z_2 \ldots, z_n$ are n complex numbers, then

$$z_1z_2\ldots z_n = r_1r_2\ldots r_n\{\cos(\theta_1+\theta_2+\ldots+\theta_n) + i\sin(\theta_1+\theta_2+\ldots+\theta_n)\} \tag{9.9}$$

Taking $z_1 = z_2 = z_3 = \ldots = z_n = z$, and
$\theta_1 = \theta_2 = \ldots = \theta_n = \theta$, and $r_1 = r_2 = \ldots = r_n = r$ in (9.9), we get

$$z^n = r^n(\cos n\theta + i\sin n\theta), n = 1, 2, 3, \ldots \tag{9.10}$$

Taking $r = 1$, and $z = \cos\theta + i\sin\theta$ in (9.10) we have

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta, \quad \text{for} \quad n = 1, 2, 3, \ldots \tag{9.11}$$

(9.11) gives the important *De Moivre's formula* for positive integral exponents n. The above formula (9.11) is evidently true for $n = 0$. Also, since

$$\begin{aligned}
z^{-1} &= \frac{1}{z}, \\
(\cos\theta + i\sin\theta)^{-1} &= \frac{1}{\cos\theta + i\sin\theta} \\
&= \frac{\cos\theta - i\sin\theta}{(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)} \\
&= \cos(-\theta) + i\sin(-\theta)
\end{aligned} \tag{9.12}$$

Hence, formula (9.11) is valid for $n = -1$. Let n be a negative integer, say $n = -m$, where $m > 1$. Then $(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m} = \left\{(\cos\theta + i\sin\theta)^{-1}\right\}^m$ $= \{\cos(-\theta) + i\sin(-\theta)\}^m$ [from (9.12)]

Hence

$$(\cos\theta + i\sin\theta)^n = \cos(-m\theta) + i\sin(-m\theta),$$

since m is a positive integer.

Therefore, (9.11) holds for $n = -m, m$ a positive integer.

In other words,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta, \quad \text{for} \quad n = \pm 1, \pm 2, \pm 3, \ldots \tag{9.13}$$

Thus, De Moivre's formula is valid for all integral values of n, positive, zero, and negative.

This formula can be extended to any real number (rational or irrational), but proving that is beyond the scope of this book.

We are now in a position to find the nth root of a given complex number for a positive integer n. Let

$$a = \rho(\cos\psi + i\sin\psi)$$

As the case $n = 1$ is trivial, we take $n \geq 2$. We have to find a complex number $z = r(\cos\theta + i\sin\theta)$ such that

$$z^n = a$$

i.e.
$$r^n(\cos n\theta + i\sin n\theta) = \rho(\cos\psi + i\sin\psi), \tag{9.14}$$

If we take $r^n = \rho$ and $n\theta = \psi$, the equation (9.14) will definitely be satisfied. Thus $r^n = \rho$ and $\theta = \dfrac{\psi}{n}$

Hence we get a root

$z = \sqrt[n]{\rho}\left(\cos\dfrac{\psi}{n} + i\sin\dfrac{\psi}{n}\right)$, where $\sqrt[n]{\rho}$ denotes the positive nth root of the positive real number ρ. But the equation (9.14) is valid, as observed earlier when we add an integral multiple of 2π to ψ. That is to say that

$$n\theta = \psi + k.2\pi, \quad \text{where} \quad k \quad \text{is an integer}$$

i.e.
$$\theta = \frac{\psi}{n} + k.\frac{2\pi}{n}$$

Hence,

$$z = \sqrt[n]{\rho}\left[\cos\left(\frac{\psi}{n} + k.\frac{2\pi}{n}\right) + i\sin\left(\frac{\psi}{n} + k.\frac{2\pi}{n}\right)\right] \tag{9.15}$$

where $k = 0, 1, 2, \ldots, (n-1)$,

For only these values of k will give different values of z and other integral values, positive or negative, of k will only repeat the values obtained by taking $k = 0, 1, 2, \ldots, (n-1)$. Note further that if $\sqrt[n]{\rho}(\cos\theta_1 + i\sin\theta_1)$ is an nth root of a, then $\theta_1 = \psi_1 + \frac{2k\pi}{n}$ for some $k = 0, 1, 2, \ldots, n-1$, where $\psi_1 = \dfrac{\psi}{n}$

Thus, the number of the nth root of a non-zero complex number is n and the modulus of each of these roots is the same non-negative real number. The arguments of these n roots are equally spaced in the sense that if ψ is the principal value of arg a i.e.

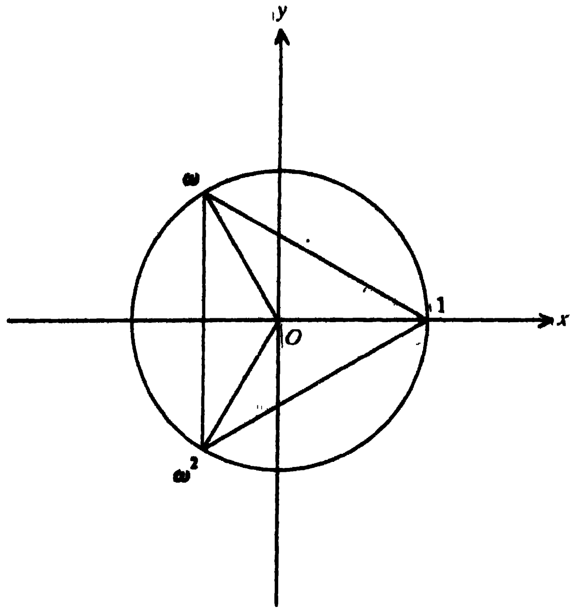


Fig. 9.8

$-\pi < \psi \le \pi$, then the arguments of other roots of a are obtained by adding respectively $\frac{2\pi}{n}, \frac{4\pi}{n} \ldots$ to $\frac{\psi}{n}$.

Let us apply this result to find the square root of a complex number $a = \rho(\cos\psi + i\sin\psi)$, $-\pi < \psi \le \pi$. Using the result (9.15), the required square roots of a, are $\sqrt{\rho}(\cos\frac{\psi}{2} + i\sin\frac{\psi}{2})$ and $\sqrt{\rho}\left\{\cos\left(\frac{\psi}{2}+\pi\right) + i\sin\left(\frac{\psi}{2}+\pi\right)\right\}$. The second root can be written as $-\sqrt{\rho}(\cos\frac{\psi}{2} + i\sin\frac{\psi}{2})$.

If $a = 1$, then $\rho = 1$ and $\psi = 0$. Hence the two square roots are 1 and -1.

The cube roots and fourth roots of 1 are of special interest. Since $1 = 1 \times (\cos 0 + i\sin 0)$, the three cube roots of 1 are $\sqrt[3]{1}(\cos\frac{0}{3} + i\sin\frac{0}{3})$, $\sqrt[3]{1}[\cos(\frac{0}{3} + \frac{2\pi}{3}) + i\sin(\frac{0}{3} + \frac{2\pi}{3})]$ and $\sqrt[3]{1}[\cos(\frac{0}{3} + \frac{4\pi}{3}) + i\sin(\frac{0}{3} + \frac{4\pi}{3})]$. Hence the three cube roots of 1 are 1, $-\frac{1}{2} + i\frac{\sqrt{3}}{2}$ and $-\frac{1}{2} - i\frac{\sqrt{3}}{2}$. All these three roots lie on the circumference of the unit circle, as shown in Fig. 9.8, and the angles between the radius vectors of the first and the second, the second and the third, and the third and the first are each $\frac{2\pi}{3}$ radians or 120°. Hence, if these points are joined by straight lines, they will form the three vertices of an equilateral triangle. If we denote the second root by ω i.e. $\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$, the third root will

be ω^2 as can be seen by actual computation, since

$$\omega^2 = \left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^2 = \left(\frac{1}{4}-\frac{3}{4}-2i\frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right)$$
$$= -\frac{1}{2}-i\frac{\sqrt{3}}{2}$$

which is the third root. Hence the three roots are 1, ω and ω^2.

The fact that the third root is the square of the second root can also be seen without any computation, because $\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$ and the third root is $\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$ which can be written as $(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})^2$ by using De Moivre's formula.

It can be easily seen by actual addition that $1+\omega+\omega^2 = 0$. In other words, *the sum of the three cube roots of unity is zero*. This result can also be deduced from the following identity:

For any complex number $z \neq 1$, we have

$$1+z+z^2 = \frac{1-z^3}{1-z} \tag{9.16}$$

The identity (9.16) can be immediately deduced from the following simple result: For $z \neq 1$,

$$(1-z)(1+z+z^2) = 1+z+z^2-z-z^2-z^3 = 1-z^3.$$

If we set $z = \omega$, the second cube root of unity, then

$$1+\omega+\omega^2 =$$

If in relation (9.15), we set $n = 4, \rho = 1$ and $\psi = 0$, we get the four roots of unity. These four roots are:

$$1,\quad \cos\frac{2\pi}{4}+i\sin\frac{2\pi}{4},\quad \cos\frac{4\pi}{4}+i\sin\frac{4\pi}{4} \text{ and } \cos\frac{6\pi}{4}+i\sin\frac{6\pi}{4}.$$

or

$$1,\quad \cos\frac{\pi}{2}+i\sin\frac{\pi}{2},\quad \cos\frac{2\pi}{2}+i\sin\frac{2\pi}{2}, \text{ and } \cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}.$$

If we set $\omega = \cos\frac{\pi}{2}+i\sin\frac{\pi}{2}$, then the third and the fourth roots are respectively $(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2})^2$ and $(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2})^3$. Thus the four roots are $1, \omega, \omega^2$ and ω^3, where $w = \cos\frac{\pi}{2}+i\sin\frac{\pi}{2} = i$. Hence the four roots are $1, i, i^2$ and i^3 or the four roots are $1, i. -1$, and $-i$. It is easily seen from the values of the four roots that their sum is zero. The four fourth roots of unity form the vertices of a square all lying on the unit circle as shown in Fig. 9.9.

Remark

It may be noted that the symbol ω has been used in the preceding discussion to denote first a cube root of unity, and later a fourth root of unity and these two roots are different.
If we take $\rho = 1$ and $\psi = 0$ in relation (9.15), we get the nth roots of 1. These n roots are $1, \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$,

$$\cos 2\cdot\frac{2\pi}{n} + i\sin 2\cdot\frac{2\pi}{n},$$
$$\cos 3\cdot\frac{2\pi}{n} + i\sin 3\cdot\frac{2\pi}{n}\ldots,$$
and $\cos(n-1)\frac{2\pi}{n} + i\sin(n-1)\frac{2\pi}{n}$.

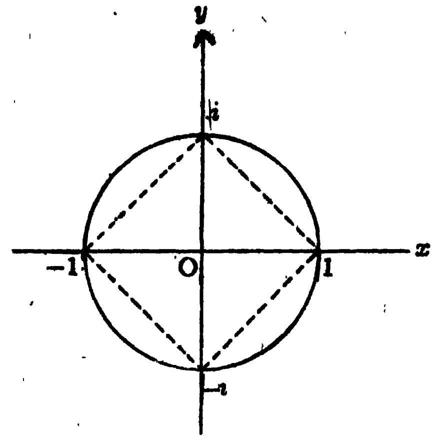


Fig 9.9

Again, if we set $\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$, then the n roots are $1, \omega, \omega^2, \omega^3, \ldots, \omega^{n-1}$, by the De Moivre formula. As stated in the beginning of this remark, the use of ω here is just for notational convenience.

It is easily seen that the nth roots of unity all lie on the unit circle and form the vertices of a regular polygon of n sides.

Example 9.6
Express $3+4i$ in polar form.

Solution

$$\begin{aligned} 3+4i &= 5\left(\frac{3}{5}+\frac{4}{5}i\right) \\ &= 5(\cos\theta + i\sin\theta) \end{aligned}$$

where θ is such that $\tan\theta = \frac{4}{3}$ as shown in Fig. 9.10. Note that $r = 5$ here.

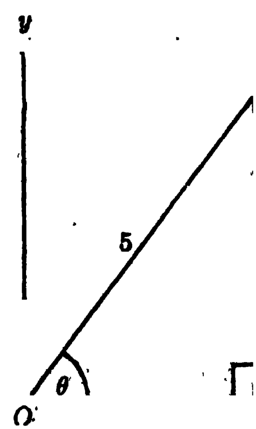


Fig 9.10

Example 9.7
Express $\sin 30° + i\cos 30°$ in polar form.

Solution
$\sin 30° = \cos 60°$ and $\cos 30° = \sin 60°$, so that

$$\begin{aligned} \sin 30° + i\cos 30° &= \cos 60° + i\sin 60° \\ &= 1(\cos 60° + i\sin 60°) \end{aligned}$$

Hence, $r = 1$ and $\theta = \frac{\pi}{3}$ The polar form is therefore $1\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

Example 9.8

If z is an arbitrary complex number, where does it lie in the Argand plane under the condition $|z + 2 - i| = 4$?

Solution

Let $z = x + iy$,

Therefore, $|z + 2 - i| = 4$

or, $|x + iy + 2 - i| = 4$

or, $\sqrt{(x+2)^2 + (y-1)^2} = 4$

or, $(x+2)^2 + (y-1)^2 = 16$

which is the equation of a circle with centre $(-2, 1)$ and radius 4.

Thus z lies on a circle with centre $-2 + i$ and radius 4.

EXERCISE 9.2

1. If two complex numbers z_1, z_2 are such that $|z_1| = |z_2|$, is it then necessary that $z_1 = z_2$?
2. If z_1, z_2, z_3 are three complex numbers such that there exists an z with $|z_1 - z| = |z_2 - z| = |z_3 - z|$ show that z_1, z_2, z_3 lie on a circle in the plane diagram.
3. Show that $\arg \bar{z} = 2\pi - \arg z$ for non-real z.

MISCELLANEOUS EXERCISE ON CHAPTER 9

1. Show that for $z \in \mathbf{C}$, $|z| = 0$ if and only if $z = 0$.
2. Show that the cube roots of unity lie on the unit circle and divide the circumference into three equal parts starting from $z = 1$.
3. State and establish a property analogous to that in question 2 above for fourth roots.
4. If P_1, P_2 are points corresponding to $z_1, z_2 \in \mathbf{C}$ in the Argand diagram, show that $P_1P_2 = |z_1 - z_2|$.
5. Simplify: (i) $(3 + 2i)(2 - i)$, (ii) $\frac{2-3i}{4-i}$
6. If z_1, z_2 are $1 - i$, $-2 + 4i$, respectively, find $\operatorname{Im}\left(\frac{z_1 z_2}{\bar{z}_1}\right)$
7. Find the square roots of $-15 - 8i$.

8. Show that $\left|\frac{z-3}{z+3}\right| = 2$ represents a circle.

9. Explain the fallacy: $-1 = i.i = \sqrt{-1}\cdot\sqrt{-1}$

$$= \sqrt{(-1)(-1)} = \sqrt{1} = 1$$

10. If $z = x + iy$, x, y real, prove that

$|x| + |y| \leq \sqrt{2}|z|$.

11. Prove that

$$\begin{aligned} \text{Re}\,(z_1 z_2) &= \text{Re}\, z_1 \,\text{Re}\, z_2 - \text{Im}\, z_1 \,\text{Im}\, z_2 \\ \text{Im}\,(z_1 z_2) &= \text{Re}\, z_1 \,\text{Im}\, z_2 + \text{Im}\, z_1 \,\text{Re}\, z_2 \end{aligned}$$

12. If ω, ω^2 be imaginary cube roots of unity, then prove that

$(3 + 3\omega + 5\omega^2)^6 - (2 + 6\omega + 2\omega^2)^3 = 0$

13. Find the value of

(a) ω^{18} (b) ω^{20} (c) ω^{-30} (d) ω^{-105}

14. Prove that

$(2-\omega)(2-\omega^2)(2-\omega^{10})(2-\omega^{11}) = 49$

15. Perform the indicated operation and find the result in the form $a + ib$

(i) $\dfrac{2-\sqrt{-25}}{1-\sqrt{-16}}$

(ii) $\dfrac{3-\sqrt{-16}}{1-\sqrt{-9}}$

16. Find the multiplicative inverse of the following:

(i) $3 + 2i$

(ii) $(2 + \sqrt{3}i)^2$

17. Find the square roots of the following:

(a) $7 - 24i$

(b) $5 - 12i$

(c) $-2 + 2\sqrt{3}\,i$

CHAPTER 10

Quadratic Equations

10.1 Solution of Quadratic Equations

You have studied polynomials in lower classes.

$$3x^2 + 2x - 1, 5x^3 - 3x + 2, \quad 4x^5 - 7x^4 - 6x^3 + 2x^2 + 3x + 6, \quad \text{etc.}$$

are examples of polynomials. A function f defined by

$$f(x) = a_0 + a_1x^2 + \ldots + a_nx^n, \quad x \in \mathbf{R}$$

where $a_0, a_1, \ldots\ldots, a_n \in \mathbf{R}$ is called a *polynomial* of a real variable with real coefficients. If $a_n \neq 0$, it is said to be of degree n. If x is assumed to be a varying complex number and $a_0, a_1, \ldots\ldots, a_n \in \mathbf{C}$, so that the function $f : \mathbf{C} \to \mathbf{C}$, it is called a complex polynomial or a polynomial of a complex variable with complex coefficients. In the last chapter we were concerned with the polynomial $x^2 + 1$ and the corresponding equation $x^2 + 1 = 0$ which we noticed cannot be satisfied by any real x. This is a polynomial of second degree. Generally, we call a polynomial of the second degree a *quadratic polynomial*. Any equation $f(x) = 0$, where f is a quadratic polynomial, is called a *quadratic equation*. The general form of a quadratic equation is

$$ax^2 + bx + c = 0 \tag{10.1}$$

where a, b, c are real numbers and $a \neq 0$. Note that if $a = 0$, then (10.1) becomes $bx + c = 0$, and this is *not* a quadratic equation because there is no term in x^2.

For a first degree (i.e. linear) equation

$$bx + c = 0, \quad (b \neq 0)$$

the root obviously is $x = -\frac{c}{b}$. This is the general solution of $bx + c = 0$ because it gives the root in terms of the coefficients (b and c) of the equation.

Similarly, if we can express the roots of (10.1) in terms of its coefficients a, b and c, we shall have found the general solution of (10.1).

You will be interested to know that the general solution of the quadratic equation was known in India at least from Brahmagupta's time about 628 A.D.

We find the general solution as given by the Indian Mathematician Shreedhara in the year 750 A.D.

Multiplying (10.1) by $4a$, we get

$$4a^2x^2 + 4abx + 4ac = 0$$

i.e.
$$(2ax + b)^2 - b^2 + 4ac = 0$$

or
$$(2ax + b)^2 = b^2 - 4ac \tag{10.2}$$

We let Δ denote the number $b^2 - 4ac$. If $\Delta \geq 0$, then taking square roots of both sides of (10.2), we get

$$2ax + b = \pm\sqrt{b^2 - 4ac}$$

which gives

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{10.3}$$

This shows that when $\Delta > 0$, (10.1) has two distinct real roots x_1 and x_2 where

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \qquad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \tag{10.4}$$

If $\Delta = 0$, then (10.2) gives

$$2ax + b = 0$$

so
$$x = -\frac{b}{2a} \tag{10.5}$$

This shows that if $\Delta = 0$, (10.1) has exactly one real root. Finally, suppose $\Delta < 0$, then $b^2 - 4ac < 0$, so $4ac - b^2 > 0$. In this case, we know from Chapter 9 that the complex numbers $\omega_1 = i\sqrt{4ac - b^2}$, $\omega_2 = -i\sqrt{4ac - b^2}$ are such that $\omega_1{}^2 = b^2 - 4ac = \omega_2{}^2$ and that no other complex number z is such that $z^2 = b^2 - 4ac$. Note here that $\sqrt{4ac - b^2}$ is the unique positive square root of the positive real number $4ac - b^2$.

From (10.2) it follows that

$$2ax + b = \omega_1 \quad \text{or} \quad \omega_2$$

so that
$$x = \frac{-b + \omega_1(\text{or } \omega_2)}{2a} = \frac{-b \pm i\sqrt{4ac - b^2}}{2a}$$

gives the solutions of (10.1)

Hence, if $\Delta < 0$, (10.1) has two distinct complex roots, namely

$$z_1 = \frac{-b + i\sqrt{4ac - b^2}}{2a}, \qquad z_2 = \frac{-b - i\sqrt{4ac - b^2}}{2a} \tag{10.6}$$

Note that the two complex roots are conjugates of each other (i.e. $z_2 = \bar{z}_1$ and $z_1 = \bar{z}_2$).

Since it is Δ which decides the number of the roots of (10.1) and also whether the roots will be real or not, the number $\Delta = b^2 - 4ac$ is called the *discriminant* of the quadratic equation (10.1). Shreedhara (in the year 750 A.D.) gave the method of solving $ax^2 + bx = c$ elegantly in just one *shloka* as follows:

चतुराहतवर्गसमैः रूपैः पक्षद्वयं गुणयेत्।
अव्यक्तवर्ग रूपैयुक्तौ पक्षौ ततो मूलम्।।

This states: the equation ($ax^2 + bx = c$) should be multiplied by $4a$, and b^2 should be added to both sides, and then square roots should be extracted.

Returning to the roots of (10.1), we note that except when $\Delta = 0$, (10.1) has two roots. Sometimes, we say that even when $\Delta = 0$, the equation has two roots but in this case both the roots are equal (equal to $-\frac{b}{2a}$) or that the root is repeated. With this understanding we can say that (10.1) has always two roots, which are (i) real and distinct if $\Delta > 0$, (ii) real and equal if $\Delta = 0$ and (iii) complex conjugates if $\Delta < 0$.

Finally, note that the sum of the two roots of (10.1) is, in any case, $-\frac{b}{a}$, because

when $\qquad \Delta > 0: \dfrac{-b + \sqrt{b^2 - 4ac}}{2a} + \dfrac{-b - \sqrt{b^2 - 4ac}}{2a} = -\dfrac{b}{a},$

when $\qquad \Delta = 0: \left(-\dfrac{b}{2a}\right) + \left(-\dfrac{b}{2a}\right) = -\dfrac{b}{a}$

and

when $\qquad \Delta < 0: \dfrac{-b + i\sqrt{4ac - b^2}}{2a} + \dfrac{-b - i\sqrt{4ac - b^2}}{2a} = -\dfrac{b}{a}$

Also, the product of the two roots is $\frac{c}{a}$.

When $b^2 - 4ac > 0$

$$\left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)$$

$$= \left(-\frac{b}{2a}\right)^2 - \frac{\left(\sqrt{b^2 - 4ac}\right)^2}{4a^2}$$

$$= \frac{b^2}{4a^2} - \frac{\left(b^2 - 4ac\right)^2}{4a^2} = \frac{c}{a}$$

When $b^2 - 4ac = 0$

$$\left(-\frac{b}{2a}\right)\left(-\frac{b}{2a}\right) = -\frac{b^2}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}$$

Verify similarly that when $b^2 - 4ac < 0$

$$\left(\frac{-b + i\sqrt{4ac - b^2}}{2a}\right)\left(\frac{-b - i\sqrt{4ac - b^2}}{2a}\right) = \frac{c}{a}$$

Example 10.1

Solve the equation $x^2 - x - 12 = 0$ using the general expression for the roots of a quadratic equation.

Solution

Here $\Delta = b^2 - 4ac = (-1)^2 - 4.1(-12) = 49 > 0$. The two real roots of the equation are, therefore, given by

$$\frac{-b) \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{49}}{2} = \frac{1 \pm 7}{2} = 4 \text{ or } -3$$

Hence, the roots are 4 and -3.

Example 10.2

Does the equation $2x^2 - 4x + 3 = 0$ have real roots? Find the roots.

Solution

Here $\Delta = (-4)^2 - 4.2.3 = -8 < 0$. The equation has, therefore, no real roots. The complex roots are

$$\frac{4 \pm \sqrt{8}i}{4} = \frac{4 \pm 2\sqrt{2}i}{4} = 1 \pm \frac{1}{\sqrt{2}}i$$

10.2 Symmetric Functions of Roots

Assume now, that two numbers α and β are given. Clearly, the equation, having α and β as its roots, is

$$(x-\alpha)(x-\beta) = 0$$

i.e.

$$x^2 - (\alpha+\beta)x + \alpha\beta = 0$$

Of course, any equation

$$a(x-\alpha)(x-\beta) = 0$$

for non-zero a also has the roots α, β. Thus a quadratic equation whose roots are numbers α, β is given by

$$x^2 - (\alpha+\beta)x + \alpha\beta = 0$$

or

$$x^2 - \text{ (sum of the roots) } x + \text{ product of the roots } = 0.$$

It is sometimes possible, without explicitly solving for the roots of a given quadratic equation, to obtain another quadratic equation whose roots have some prescribed relations. We shall illustrate this by an example. Let α, β be the roots of the quadratic equation

$$ax^2 + bx + c = 0 \tag{10.7}$$

we want to obtain an equation whose roots are twice the roots of (10.7). If α', β' are the roots of the required equation, we have

$$\alpha' + \beta' = 2\alpha + 2\beta = 2(\alpha+\beta) = -\frac{2b}{a}$$

and

$$\alpha'\beta' = (2\alpha)(2\beta') = 4\,\alpha\beta = 4\frac{c}{a}$$

Hence, the required equation is

$$x^2 + \frac{2b}{a}x + \frac{4c}{a} = 0 \quad \text{or} \quad ax^2 + 2bx + 4c = 0$$

Let us now find a quadratic equation whose roots are squares of the roots of (10.7). If α', β' are the roots of the required equation. $\alpha' = \alpha^2$ and $\beta' = \beta^2$. Hence,

$$\alpha' + \beta' = \alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = \frac{b^2}{a^2} - \frac{2c}{a} = \frac{b^2 - 2ca}{a^2}$$

$$\alpha'\beta' = \alpha^2\beta^2 = (\alpha\beta)^2 = \frac{c^2}{a^2}$$

The equation having these roots is, therefore,

$$x^2 - \frac{b^2 - 2ca}{a^2}x + \frac{c^2}{a^2} = 0$$

or

$$a^2x^2 - (b^2 - 2ca)x + c^2 = 0$$

Note that in the above examples the sum of the roots of the desired equations is a *symmetric* function of the roots of the original equation in the sense that the expression is not affected by interchanging the roots. For instance, if α, β are the roots of (10.7), as earlier,

$$\alpha^2 + \beta^2, \alpha^3 + \beta^3, \alpha^2\beta + \alpha\beta^2, \frac{1}{\alpha} + \frac{1}{\beta} \text{ and } \frac{1}{\alpha^2} + \frac{1}{\beta^2}$$

are all symmetric functions of α, β. All these functions can be expressed in terms of the symmetric functions $\alpha + \beta$ and $\alpha\beta$.
For,

$$\begin{aligned}
\alpha^2 + \beta^2 &= (\alpha+\beta)^2 - 2\alpha\beta \\
\alpha^3 + \beta^3 &= (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta) \\
\alpha^2\beta + \alpha\beta^2 &= \alpha\beta(\alpha+\beta) \\
\frac{1}{\alpha} + \frac{1}{\beta} &= \frac{\alpha+\beta}{\alpha\beta} \\
\frac{1}{\alpha^2} + \frac{1}{\beta^2} &= \frac{(\alpha+\beta)^2 - 2\alpha\beta}{(\alpha\beta)^2}
\end{aligned}$$

This shows how we can, without actually solving equation (10.7), obtain a quadratic equation whose roots are any one pair of the following pairs of numbers:

$$\alpha^2, \beta^2 \; ; \; \alpha^3, \beta^3 \; ; \; \alpha^2\beta, \alpha\beta^2 \; ; \; \frac{1}{\alpha}, \frac{1}{\beta}; \frac{1}{\alpha^2}, \frac{1}{\beta^2}.$$

Example 10.3

If α, β are the roots of the equation

$$3x^2 - 2x - 6 = 0$$

find $\alpha^3 + \beta^3$.

Solution

Since α, β are given to be tge roots of $3x^2 - 2x - 6 = 0$

$$\therefore \quad \alpha + \beta = \frac{2}{3} \quad \text{and} \quad \alpha\beta = -2.$$

Since $(\alpha + \beta)^3 = \alpha^3 + \beta^3 + 3\alpha\beta(\alpha + \beta)$, we have

$$\begin{aligned} \alpha^3 + \beta^3 &= (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta) \\ &= (\alpha + \beta)[(\alpha + \beta)^2 - 3\alpha\beta] \\ &= \frac{2}{3}\left[\left(\frac{2}{3}\right)^2 - 3(-2)\right] = \frac{116}{27} \end{aligned}$$

Example 10.4

Find p if the sum of the squares of the roots of the equation $x^2 + px - 3 = 0$ is equal to 10.

Solution

If α, β are the roots of the given equation, we have $\alpha + \beta = -p$, $\alpha\beta = -3$

Now

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = p^2 + 6$$

$$\begin{aligned} \text{But} \qquad \alpha^2 + \beta^2 &= 10 \quad \text{(given).} \\ \therefore \quad p^2 + 6 &= 10 \\ \text{i.e.} \qquad p^2 &= 4 \\ \text{or} \qquad p &= \pm 2 \end{aligned}$$

10.3 Graph of a Quadratic Polynomial

We shall briefly study the graph of a quadratic polynomial i.e. the curve whose equation is $y = ax^2 + bx + c$, $(a \neq 0)$.

Multiplying by $4a$ and completing the square on the right as before, the equation becomes

$$4ay = (2ax + b)^2 - (b^2 - 4ac)$$

or

$$4ay + \Delta = (2ax + b)^2$$

which, on division by $4a$, noting that $a \neq 0$, becomes

$$y + \frac{\Delta}{4a} = a\left(x + \frac{b}{2a}\right)^2$$

If we now shift the origin to the point $(-\frac{b}{2a}, -\frac{\Delta}{4a})$, i.e. write $X = x + \frac{b}{2a}$, $Y = y + \frac{\Delta}{4a}$, this equation becomes (in the new coordinates X, Y),

$$Y = aX^2,$$

which as we know is the equation of a parabola. Thus a quadratic polynomial always represents a parabola.

If α is a real root of $ax^2 + bx + c = 0$, then $a\alpha^2 + b\alpha + c = 0$, so that the point $(\alpha, 0)$ lies on $y = ax^2 + bx + c$; every real root of $ax^2 + bx + c = 0$ represents a point of intersection of the parabola with the x-axis. Conversely, if the parabola $y = ax^2 + bx + c$ intersects the x-axis at a point $(p, 0)$ then $0 = ap^2 + bp + c$, so that p is the real root of $ax^2 + bx + c = 0$. Thus, the intersection of the parabola $y = ax^2 + bx + c$ with the x-axis

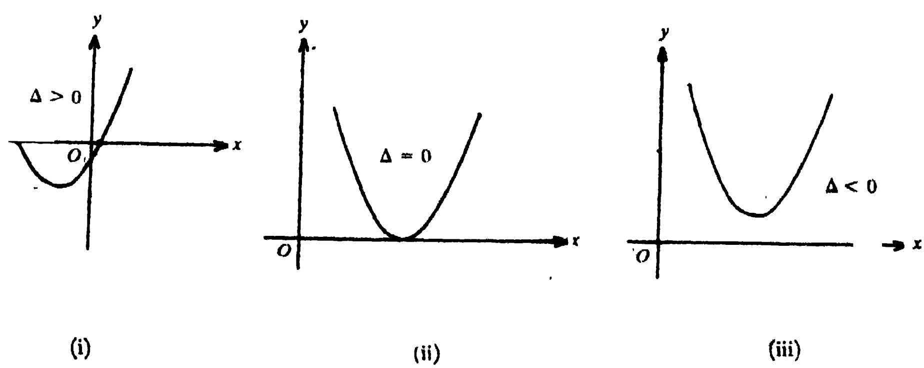


Fig. 10.1

give us all the real roots of the equation (10.1). We can, therefore, say that if $\Delta > 0$, the parabola will intersect the x-axis in two distinct points; if $\Delta = 0$, the parabola will just touch the x-axis at one point and if $\Delta < 0$, the parabola will not intersect the x-axis at all. (Fig. 10.1).

10.4 Applications

In this section we shall illustrate some of the results presented in the earlier sections of the chapter. We also make use of quadratic equations in solving some problems.

Example 10.5

The roots α, β of the equation $x^2 - 3ax + a^2 = 0$ are such that $\alpha^2 + \beta^2 = 1.75$. Find the value of a.

Solution

Here $\alpha + \beta = 3a$, $\alpha\beta = a^2$ so that
$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = (3a)^2 - 2a^2 = 7a^2$.
Thus $7a^2 = 1.75$ which means that $a^2 = 0.25$ so that $a = \pm 0.5$.

Sometimes quadratic equations are of use in solving polynomial equations of higher degree, as in the case of the following examples:

Example 10.6

Solve $(x^2 - 5x)^2 - 30(x^2 - 5x) - 216 = 0$.

Solution

This is an equation of degree 4.
Set $x^2 - 5x = y$. Then we have
$y^2 - 30y - 216 = 0$,
a quadratic equation in the unknown y. By factorising or otherwise the two solutions of this equation are seen to be $y = -6$ or 36. Since y stands for $x^2 - 5x$, these solutions, in turn, give rise to two quadratic equations:

(a) $y = x^2 - 5x = -6$ or $x^2 - 5x + 6 = 0$

and (b) $y = x^2 - 5x = 36$ or $x^2 - 5x - 36 = 0$.

Both these equations are solved easily by factorising the corresponding quadratic expressions. Thus we have four solutions for x, i.e. 2, 3, -4, 9.

Example 10.7

Solve the biquadratic (a fourth degree) equation

$$x^4 - 8x^2 - 9 = 0.$$

Solution

Set $x^2 = y$. The equation then reduces to

$$y^2 - 8y - 9 = 0$$

which is easily solved to get two values of y, viz. $y = 9$ or -1
i.e. $x^2 = 9$ or -1 so that $x = \pm 3$ or $\pm i$.

Quadratic equations may be useful in solving simultaneous equations too, as illustrated in the following example.

Example 10.8

Solve the system of equations

$$(x + y)^2 - 2(x + y) = 15 \tag{10.8}$$

$$xy = 6 \tag{10.9}$$

Solution

Putting $x + y = u$, (10.8) becomes

$$u^2 - 2u - 15 = 0 \tag{10.10}$$

(10, 10) is easily solved and we get $u = 5$ or -3. This means that we have two systems of equations:

$$\text{(A):}\quad \begin{aligned} x + y &= 5 \\ xy &= 6 \end{aligned} \qquad \text{and} \qquad \text{(B):}\quad \begin{aligned} x + y &= -3 \\ xy &= 6 \end{aligned}$$

Eliminating y in (A), we get

$$\begin{aligned} & x(5 - x) = 6 \\ \text{or}\quad & x^2 - 5x + 6 = 0 \end{aligned}$$

which gives $x = 3$ or 2.

When $x = 3, y = 2$. When $x = 2, y = 3$. Thus $x = 3, y = 2$; $x = 2, y = 3$ are solutions of the given equations.

Similarly from (B), we have

$$x = \frac{-3 + i\sqrt{5}}{2},\ y = \frac{-3 - i\sqrt{5}}{2} \quad \text{and } x = \frac{-3 - i\sqrt{5}}{2},\ y = \frac{-3 + i\sqrt{5}}{2}$$

are also solutions of the given equations.

Hence required roots are: $x = 2, y = 3$; $x = 3, y = 2$; $x = \dfrac{-3 + i\sqrt{15}}{2}$, $y = \dfrac{-3 - i\sqrt{15}}{2}$;

$$x = \frac{-3 - i\sqrt{15}}{2}, y = \frac{-3 + i\sqrt{15}}{2}$$

Some problems can be transformed into equations whose solution may depend on quadratic equations as illustrated in the following example.

Example 10.9

A two-digit number is four times the sum and three times the product of its digits. Find the number.

Solution

If x is the digit in the ten's place and y in the unit's place of the number, then the number is $10x + y$. We have now

$$10x + y = 4(x + y)$$
$$10x + y = 3xy$$

as the system of equations arising out of the data of the problem. The first of these equations is

$$6x = 3y \text{ or } 2x = y$$

Substituting in the second we have

$$10x + 2x = 6x^2.$$

This gives $x = 0$ or $x = 2$. If $x = 0$, then $y = 0$ and the number in this case does not possess two digits The solution is, therefore, $x = 2, y = 4$, i.e. the number desired is 24.

Example 10.10

A swimming pool is fitted with three pipes with uniform flow. The first two pipes operating simultaneously, fill the pool in the same time during which the pool is filled by the third pipe alone. The second pipe fills the pool five hours faster than the first pipe and four hours slower than the third pipe. Find the time required by each pipe to fill the pool individually.

Solution

Let V be the volume of the pool and x the number of hours required by the second pipe alone to fill the pool. Then the parts of the pool filled by the first, second and third pipes in one hour are respectively

$$\frac{V}{x+5}, \frac{V}{x} \quad \text{and} \quad \frac{V}{x-4}$$

It is given that
$$\frac{V}{x+5} + \frac{V}{x} = \frac{V}{x-4}$$

Since $V \neq 0$, this means that
$$\frac{1}{x+5} + \frac{1}{x} = \frac{1}{x-4}$$

giving rise to the quadratic equation $x^2 - 8x - 20 = 0$. Hence, $x = 10$ or -2. But the negative solution does not fit the problem. So $x = 10$. Thus, the time required by the three pipes to fill the pool is 15, 10, 6 hours, respectively.

EXERCISE 10.1

1. Obtain a quadratic equation whose roots are $\alpha = 2, \beta = 3$.
2. Without computing the roots α, β of $3x^2 + 2x + 6 = 0$, find (i) $\frac{1}{\alpha} + \frac{1}{\beta}$, (ii) $\alpha^2 + \beta^2$, (iii) $\alpha^3 + \beta^3$.
3. Solve the equation $\sqrt{x} = x - 2$ in $\mathbf{C}$.
4. Solve the equation $\sqrt{3x+1} - \sqrt{x-1} = 2$ in $\mathbf{C}$.
5. For what values of a is one of the roots of the equation
$$x^2 + (2a+1)x + a^2 + 2 = 0$$
twice the value of the other?
6. A piece of cloth costs Rs 35.00. If the piece were 4m longer and each metre costs Re 1.00 less, the cost would remain unchanged. How long is the piece?
7. A group of students decided to buy a tape-recorder from 170 to 195 rupees. But at the last moment two students backed out of the decision so that the remaining students had to pay 1 rupee more than they had planned. What was the price of the tape-recorder if the students paid equal shares.
8. Solve $(x^2 - 5x + 7)^2 - (x-2)(x-3) = 4$.
9. Solve the equation $\dfrac{\sqrt{a+x} + \sqrt{a-x}}{\sqrt{a+x} - \sqrt{a-x}} = \frac{a}{x}$. in $\mathbf{C}$ given that $a \in \mathbf{R}$.
10. Solve the equation
$$25x^2 - 30x + 9 = 0$$
11. A factory kept increasing its output by the same percentage every year. Find the percentage if it is known that the output doubled in the last two years.

12. A number of points are marked on a plane and are connected pairwise by a line segment. If the total number of line segments is 10, how many points are marked on the plane?
13. Solve the equation $x^2 + px + 45 = 0$, given that the square of the difference of its roots is equal to 144.
14. Find the value of a for which the roots $\alpha . \beta$ of the equation $x^2 - 6x + a = 0$ satisfy the relation $3\alpha + 2\beta = 20$.
15. Solve:
$$\frac{x-1}{x-2} - \frac{x-2}{x-3} = \frac{x-5}{x-6} - \frac{x-6}{x-7}$$
16. Solve the equation:
$$5^{x+1} + 5^{2-x} = 5^3 + 1$$
17. Solve the equation:
$$\sqrt{\frac{x}{1-x}} + \sqrt{\frac{1-x}{x}} = \frac{13}{6}$$
18. For what values of k,
$(4-k)x^2 + (2k+4)x + (8k+1) = 0$, is a perfect square.
19. Show that the roots of
$x^2 - 2x(m + \frac{1}{m}) + 3 = 0$ are real for real values of m.

CHAPTER 11

Sequences and Series

11.1 Sequences

You may remember that we defined the notion of a function in Chapter 1. Denote by $\mathbf{N}_n$ the set of natural numbers $1, 2, \ldots, n$, i.e. $\mathbf{N}_n = \{1, 2, \ldots, n\}$. $\mathbf{N}$, as usual, denotes the set of all natural numbers $1, 2, 3, \ldots$. Each $\mathbf{N}_n$ has a finite number of elements, to be precise n elements, while $\mathbf{N}$ has an infinite number of elements. We define a *sequence in a non-empty set* X to be a map $f : \mathbf{N}_n \to X$ or a map $f : \mathbf{N} \to X$. If $X = \mathbf{R}$, we call the sequence f *a real sequence and if* $X = C$, we call f a *complex sequence.* If dom $f = \mathbf{N}_n$, the sequence f is said to be a *finite* sequence and if $f = \mathbf{N}$, it is called an *infinite sequence.* In this chapter, we shall be concerned only with real or complex sequences. If f is a sequence, then for any $k \in \mathbf{N}_n$ or $\mathbf{N}$ according as f is a finite sequence or an infinite sequence

$$f(k) = a_k \in X, \text{ where } X \text{ is either } \mathbf{R} \text{ or } \mathbf{C}.$$

Thus $a_1, a_2, \ldots, a_n$; or $a_1, a_2, a_3, \ldots, a_n, \ldots$ as the case may be, determine the function f or the sequence f. For, to get the definition of f, what we need precisely is the answer to the question: what is $f(k)$ for $k \in \mathbf{N}_n$ or $\mathbf{N}$ as the case may be? On this account, it is customary to call

$$a_1, a_2, \ldots, a_n$$

a finite sequence, and

$$a_1, a_2, \ldots, a_n, \ldots.$$

an infinite sequence. The following notations are in vogue:

$\{a_k\}_{k=1}^{n}$ for a finite sequence $a_1, a_2, \ldots, a_n$;

$\{a_n\}_{n=1}^{\infty}$ or simply $\{a_n\}$ for an infinite sequence $a_1, a_2, \ldots, a_n, \ldots$

Sequences following certain patterns are more often called *progressions*. In this chapter, we study such special sequences. If $a_1, a_2, \ldots, a_n$, or $a_1, a_2, \ldots, a_n, \ldots$ is a sequence, then for $k \in \mathbf{N}_n$ or $\mathbf{N}$, as the case may be, a_k is called the *kth term* of the sequence.

More often, a sequence is given with explicit formula for the kth term e.g. the finite sequence

$$2, 4, 6, 8, \ldots, 24$$

is given by $a_k = 2k$, $k \in \mathrm{N}_{12}$, i.e., $k = 1, 2, \ldots,\ 12$
The finite sequence

$$1, 2, 3, 6, 12, 24, 48, 96, 192, 384$$

is given by

$$a_1 = 1, a_2 = 2, a_3 = a_1 + a_2, a_4 = a_1 + a_2 + a_3, \ldots$$
$$a_k = a_1 + a_2 + \ldots + a_{k-1},\ k = 3, 4, 5, \ldots, 10$$

In other words, the first two terms are respectively 1, 2 and beginning from the third, any term is the sum of all its preceding terms. Incidentally, in any sequence $\{a_k\}$, a_{k-1} is the *preceding* term of a_k if $k > 1$, a_{k+1} is the *succeeding* term for $k \geq 1$, provided in the case of a finite sequence a_{k+1} is defined. a_1 is called the *first term* and in the case of the finite sequence $a_1, a_2, \ldots, a_n, a_n$ is called the *last term*. a_k, a_{k+1} are said to be *consecutive terms* if $k = 1, 2, 3, \ldots, n-1$ or $k = 1, 2, 3, \ldots$ according as the sequence is finite or infinite.

To define a sequence we need *not* always have an explicit formula for the nth term e.g. we have the infinite sequence of all prime numbers

$$2, 3, 5, 7, 11, 13, 17, \ldots$$

for which we may not be able to give an explicit formula for the nth term. Again, if we take the successive decimal approximations to the irrational number $\sqrt{2}$, we get a sequence

$$1.4, 1.41, 1.414, 1.4142, \ldots$$

whose nth term may not be given by a formula. What is important is the rule which defines the nth term though it may not lend itself to an explicit formula for the term.

11.2 Arithmetic Progression (A.P.)

A finite or infinite sequence

$$a_1, a_2, \ldots, a_n$$

or

$$a_1, a_2, \ldots, a_n, \ldots.$$

is said to be an *arithmetic progression* (abbreviated as A.P.)
if

$a_k - a_{k-1} = d$, a constant independent of k , for $k = 2, 3, \ldots, n$ or $k = 2, 3, \ldots$
as the case may be. In other words, the difference between any two consecutive terms

is a constant. In the above definition, the constant d is called the *common difference* (abbreviated as C.D.) of the arithmetic progression. Now,

$$a_2 - a_1 = d, \text{ or } a_2 = a_1 + d, a_3 - a_2 = d \text{ or } a_3 = a_2 + d = a_1 + d + d = a_1 + 2d$$

We can guess that
$a_k = a_1 + (k-1)d, k = 1, 2, \ldots, n$ or $k = 1, 2, \ldots,$ as the case may be.
This is so as seen by induction: For $k = 1, a_k = a_1 + (k-1)d$ as is clear. Assume that $a_k = a_1 + (k-1)d$. Then the sequence being an A.P.,

$$\begin{aligned} a_{k+1} - a_k = d \text{ or } a_{k+1} &= a_k + d \\ a_{k+1} = a_k + d &= a_1 + (k-1)d + d \\ &= a_1 + kd \end{aligned}$$

showing that the expression for a_k holds for a_{k+1} also if it holds for a_k. Thus, by induction, the expression holds for any relevant k since it holds for a_1. Thus, in an A.P. whose first term is a_1 and common difference is d, the formula for the nth term is

$$a_n = a_1 + (n-1)d \tag{11.1}$$

Formula 11.1 shows that the nth term of an A.P. is a linear function of n, i.e. the nth term a_n is of the form

$$a_n = pn + q$$

where p and q are constants (not depending on n). In (11.1), $p = d$, $q = a_1 - d$. Conversely, if $a_1, a_2, \ldots$ is a sequence in which the nth term is a linear function of n, say,

$$a_n = pn + q$$

then the sequence is an A.P., because for every $k \geq 1$,

$$a_k - a_{k-1} = pk + q - [p(k-1) + q] = p,$$

i.e. $a_k - a_{k-1}$ is a constant.

We shall now obtain a formula for the sum S_n of the first n terms of an A.P. Now,

$$S_n = a_1 + a_2 + \ldots + a_n$$

Using the formula (11.1) for the nth term and letting $a_1 = a$, we have

$$S_n = a + (a+d) + (a+2d) + \ldots + [a + (k-1)d] + \ldots + [a + (n-1)d] \tag{11.2}$$

Reversing the order of summation in (11.2)

$$S_n = [a+(n-1)d] + [a+(n-2)d] + \ldots + [a+(n-k)d] + \ldots + (a+d) + a \tag{11.3}$$

Note that the first summand in (11.2) and the first summand in (11.3) when added give $2a+(n-1)d$, the second summand in (11.2) and the second summand in (11.3) when added give $2a+(n-1)d$. Thus, adding both sides of (11.2) and (11.3), we have

$$2S_n = 2a+(n-1)d+2a+(n-1)d+\ldots+2a+(n-1)d \tag{11.4}$$

where there are n summands in (11.4), since equal number of summands are there both in (11.2) and (11.3). (11.4) can be rewritten as

$$2S_n = n\left\{2a+(n-1)d\right\}$$

Or, the sum of the first n terms of the A.P.

$$S_n = \frac{n}{2}\left\{2a+(n-1)d\right\} \tag{11.5}$$

(11.5) can be interpreted as follows:

$$\begin{aligned} S_n &= \frac{n}{2}\left\{a+a+(n-1)d\right\} \\ &= n\left(\frac{a_1+a_n}{2}\right) \end{aligned} \tag{11.6}$$

i.e. the sum S_n of the first n terms of the A.P. = n times the mean of the first and the nth terms.

The above formulation helps to sum the terms of an A.P. starting from its kth term upto the lth term $k \leq l$.

Note that

$$a_k, a_{k+1}, \ldots, a_l$$

is itself an A.P. as seen from the definition. Thus, the formula (11.6) yields the desired sum

$$S_{kl} = (l-k+1)\left(\frac{a_k+a_l}{2}\right)$$

As the number of terms is $(l-k+1)$ and $\left(\frac{a_k+a_l}{2}\right)$ is the mean of kth term and last term i.e. the lth term, S_{kl} denotes the sum from k^{th} term to l^{th} term.

Example 11.1
Find the sum to n terms of the sequence $\{a_n\}$ where

$$a_n = 5 - 6n, n \in \mathrm{N}$$

Solution
$a_{n+1} - a_n = 5 - 6(n+1) - (5 - 6n) = -6, n \in \mathrm{N}$. Thus $\{a_n\}$ is an A.P. with first term -1 and common difference -6. So the sum S_n to n terms is given by

$$S_n = \frac{(-1 + 5 - 6n)n}{2} = \frac{n(4 - 6n)}{2} = n(2 - 3n).$$

Example 11.2
Find the nth term of the A.P.

$$5, 2, -1, -4, -7, \ldots.$$

Solution
The difference between the $(n+1)$th term and nth term is -3. So, if a_n is the nth term, then

$$\begin{aligned} a_n = a_1 + (n-1)d &= 5 + (n-1)(-3) \\ &= 8 - 3n. \end{aligned}$$

EXERCISE 11.1

1. The third term of an A.P. is 25 and the tenth term is -3. Find the first term and the common difference.
2. A sequence $\{a_n\}$ is given by

$$a_n = n^2 - 1, n \in \mathrm{N}.$$

   Show that it is *not* an A.P.
3. How many terms of the sequence 18, 16, 14, ... should be taken so that their sum is zero?
4. If the p^{th} term of an A.P. is c and the q^{th} term is d, what is the r^{th} term?
5. Split 69 into 3 parts in A.P. such that the product of the two smaller parts is 483.
6. If $a + b + c \neq 0$ and $\frac{b+c}{a}, \frac{c+a}{b}, \frac{a+b}{c}$ are in A.P., prove that $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are also in A.P.
7. For the A.P., $a_1, a_2, a_3, \ldots$, if $\frac{a_4}{a_7} = \frac{2}{3}$, find $\frac{a_6}{a_8}$.

8. Sums of the first p, q and r terms of an A.P. are a, b, c respectively. Prove that

$$\frac{a}{p}(q-r)+\frac{b}{q}(r-p)+\frac{c}{r}(p-q)=0$$

9. If the 12^{th} term of an A.P. is -13 and the sum of the first four terms is 24, what is the sum of the first 10 terms?
10. If the 5^{th} and the 12^{th} terms of an A.P. are 30 and 65 respectively, what is the sum of the first 20 terms?
11. If the first term of an A.P. is 22, the common difference is -4 and the sum to n terms is 64, find n. Explain the double answer.
12. If the p^{th} term of an A.P. is $\frac{1}{q}$ and the q^{th} term is $\frac{1}{p}$, prove that the sum of the first pq terms must be $\frac{1}{2}(pq+1)$.

11.3 Examples of A.P. and Insertion of Arithmetic Means

We shall now consider some examples of an arithmetic progression. The simplest arithmetic progression is that of the sequence N of natural numbers:

$$1, 2, 3, \ldots$$

In this case, both the first term and the common difference are 1 If S_n stands for the sum of the first n natural numbers, it follows from (11.6) that

$$S_n = n\frac{(1+n)}{2} = \frac{n(n+1)}{2} \qquad (11.7)$$

The sequence 2, 4, 6, 8, ... is, again, an arithmetic progression whose first term is 2 and the common difference is 2. The nth term a_n here is

$$a_n = 2+\overline{n-1}.2 = 2n$$

By (11.6), the sum S_n of the first n terms of this A.P. is given by

$$S_n = n\frac{2+2n}{2} = n(n+1)$$

Comparing this with (11.7) it may be noted that this is twice the sum of the first n natural numbers, as it should be.

The sequence 1, 3, 5, 7, ... of odd natural numbers is also an A.P. Here the first term is 1 and the common difference is 2. So

$$a_n = 1+\overline{n-1}.2 = 2n-1$$

Hence, S_n the sum of the first n odd natural numbers is given by

$$S_n = n\frac{1+2n-1}{2} = n^2$$

As a check

$$\begin{aligned}1+2+3+\ldots+2n &= (2+4+\ldots+2n)+(1+3+\ldots+2n-1)\\ &= n(n+1)+n^2\\ &= n(2n+1)\\ &= 2n\frac{2n+1}{2} \quad \text{(according to (11.7))}\end{aligned}$$

Given one or more A.P., we can generate more A.Ps. For example if $\{a_n\}$ and $\{b_n\}$ are two A.P s so is

$$\{a_n+b_n\}.$$

Verify this. As an illustration, $a_n=n$ and $b_n=2n$ give rise to the A.P. $\{a_n+b_n\}$ where

$$a_n+b_n=3n$$

If $a_n=n$, $b_n=2n-1$, we obtain $a_n+b_n=3n-1$ so that we have the A.P.

$$2,5,8,11,\ldots$$

with first term 2 and common difference 3. Again, by adding a constant to each term of an A.P., we obtain another A.P. By adding the constant 1 to the A.P.

$$1,3,5,7,\ldots$$

we obtain the A.P.

$$2,4,6,8,\ldots$$

It is also possible to construct an A.P. which is a finite sequence such that a given number a is its first term and another given number $b\neq a$ is its kth term for any prescribed natural number $k>1$. This would mean that there would be $k-2$ terms in between the terms a and b in the A.P. The procedure is known as *inserting $k-2$ arithmetic means between a and b*. For convenience of notation, we shall insert n arithmetic means between a and b. Assume that these means are $a_2,\ldots,a_{n+1}$

so that $$a=a_1,a_2,\ldots,a_{n+1},a_{n+2}=b$$

is an A.P. with first term a and $(n+2)$th term b. If d is the common difference of the A.P., then

$$\begin{aligned}b &= a_{n+2}=a+(n+2-1)d\\ &= a+(n+1)d\end{aligned}$$

Thus

$$d=\frac{b-a}{n+1}.$$

It is then immediate from formula (11.1) that

$$a_2 = a + \frac{b-a}{n+1}, \quad a_3 = a + 2\frac{b-a}{n+1}, \quad a_4 = a + 3\frac{b-a}{n+1}, \ldots, a_k = a + (k-1)\frac{b-a}{n+1}, \ldots,$$

$$a_{n+1} = a + n\frac{b-a}{n+1}$$

are the n arithmetic means (abbreviated as A.M.).

Example 11.3

Insert three A.M s between 3 and 19.

Solution

If $3, a_2, a_3, a_4, 19$ is the resulting A.P. whose common difference is d, then

$$\begin{aligned} 19 &= 3 + (5-1)d = 3 + 4d \\ \text{or,} \quad 4d &= 16 \text{ or } d = 4 \end{aligned}$$

Thus $a_2 = 7, a_3 = 11, a_4 = 15$ are the three A.Ms.

Example 11.4

Find the sum of all the natural numbers with two digits.

Solution

The sequence of the numbers is

$$10, 11, \ldots, 99$$

which is an A.P. with common difference 1. Hence, the sum S of these numbers is

$$S = 90.\frac{10+99}{2} = 45 \times 109 = 4905$$

Example 11.5

If the numbers a^2, b^2, c^2 are given to be in A.P., show that $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P.

Solution

The result is established, if we show that

$$\begin{aligned} \frac{1}{c+a} - \frac{1}{b+c} &= \frac{1}{a+b} - \frac{1}{c+a} \\ \text{i.e.} \quad \frac{b-a}{(c+a)(b+c)} &= \frac{c-b}{(a+b)(c+a)} \\ \text{i.e.} \quad \frac{b-a}{b+c} &= \frac{c-b}{a+b} \\ \text{i.e.} \quad b^2 - a^2 &= c^2 - b^2 \\ \text{i.e.} \quad a^2, b^2, c^2 & \quad \text{are in A.P.} \end{aligned}$$

which is the case.

Note:

It is customary to say that $(b+c), (c+a), (a+b)$ are in *harmonic progression* (H.P.) when their reciprocals $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P. More generally, $a_1, a_2, a_3, \ldots$ is an H.P. if and only if $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \ldots$ is an A.P. Problems about H.P. thus get converted immediately to those about A.P.

Example 11.6

If a, b, c are in A.P., prove that $b+c, c+a, a+b$ are also in A.P.

Solution

$b+c, c+a, a+b$ would be in A.P. if $c+a$ is the (one) A.M. between $b+c$ and $a+b$. This would be the case provided

$$c+a = \frac{1}{2}\{(b+c)+(a+b)\}$$

i.e.

$$c+a = \frac{1}{2}(2b+a+c)$$

It is given that a, b, c are in A.P., so $b = \frac{1}{2}(a+c)$, or, $2b = a+c$.
Thus, we have

$$c+a = \frac{1}{2}(a+c+a+c)$$

which is true.

Example 11.7

Solve the equation

$$1+6+11+16+\ldots+x = 148$$

Solution

1, 6, 11, 16, . . . is an A.P. with first term 1 and common difference 5. Thus the equation is equivalent to the assertion that the sum of the first k terms of this A.P. is 148 if the kth term is x. In other words,

$$\begin{aligned} S_k &= \frac{k}{2}[2\cdot 1 + (k-1)5] \\ &= \frac{k}{2}(5k-3) = 148 \end{aligned}$$

i.e. $\quad 5k^2 - 3k = 296 \quad$ or $\quad 5k^2 - 3k - 296 = 0$

i.e. $\quad 5k^2 - 40k + 37k - 296 = 0$

i.e. $\quad (5k+37)(k-8) = 0$

The only solution admissible is $k = 8$, in which case $x = 1 + (8-1)5 = 36$.

EXERCISE 11.2

1. Show that if the positive numbers a, b, c are in A.P. so are the numbers $\frac{1}{\sqrt{b}+\sqrt{c}}, \frac{1}{\sqrt{c}+\sqrt{a}}, \frac{1}{\sqrt{a}+\sqrt{b}}$.
2. Insert 6 arithmetic means between 3 and 24.
3. Find the common difference of an A.P. whose first term is 100 and the sum of whose first six terms is five times the sum of the next six terms.

11.4 Geometric Progression (G.P.)

We now turn to the study of yet another special kind of sequence, viz. a geometric progression. Let $a_1, a_2, \ldots, a_n; a_1, a_2, \ldots, a_n, \ldots$ be respectively a finite or an infinite sequence. Assume that *none* of the a_n's is 0 and that

$$\frac{a_{k+1}}{a_k} = r, \quad \text{a constant (i.e. independent of } k) \tag{11.8}$$

for $k = 1, 2, \ldots, n$ or $k = 1, 2, 3, \ldots$ as the case may be. We then call $\{a_k\}_{k=1}^{n}$ or $\{a_k\}_{k=1}^{\infty}$, as the case may be, a *geometric progression* (abbreviated as G.P.). The constant ratio r in (11.8) is called the *common ratio* (abbreviated as C.R.) of the G.P. Now (11.8) implies

$$a_{k+1} = a_k r = a_{k-1} r^2 = a_{k-2} r^3 = \ldots = a_1 r^k.$$

Thus if $a_1 = a$ is the first term of a G.P., the kth term a_k of the G.P. is

$$a_k = a_1 r^{k-1} = a r^{k-1} \tag{11.9}$$

It follows that, given the first term a and the C.R. r, the G.P. can be rewritten as

$$a, ar, ar^2, \ldots, ar^{n-1}$$

or

$$a, ar, ar^2, \ldots, ar^{n-1}, \ldots$$

according as it is finite or infinite.

Thus we have seen that the nth term of a G.P. is $a_n = ar^{n-1}$ where a is the first term and r the common ratio. Conversely, any sequence, $a_1, a_2, \ldots$ in which the nth term is given by $a_n = ar^{n-1}$ where a and r are constants, is a G.P. with first term a and common ratio r. This is because the first term $= a_1 = ar^{1-1} = a$ and for all $k \geq 2$,

$$\frac{a_{k+1}}{a_k} = \frac{ar^k}{ar^{k-1}} = r$$

Next, we obtain an expression for the sum S_n of the first n terms of a G.P.

$$a, ar, ar^2, \ldots, ar^{n-1}$$

Now,

$$S_n = a + ar + ar^2 + \ldots + ar^{n-1}$$

and

$$rS_n = ar + ar^2 + \ldots + ar^{n-1} + ar^n.$$

If $r = 1$, the sequence reduces to a constant sequence $a, a, a, \ldots$. Hence we can assume that $r \neq 1$. Now, subtracting the second sum from the first, we get

$$(1-r)S_n = a - ar^n$$

$$\text{or, } S_n = \frac{a(1-r^n)}{(1-r)} = \frac{a_1 - ra_n}{1-r} \tag{11.10}$$

where a_1 is the first term, a_n is the nth term and r is the common ratio.

If $a_1, a_2, \ldots, a_n$; or $a_1, a_2, \ldots, a_n, \ldots$ is a G.P. with positive terms, then

$$\frac{a_3}{a_2} = \frac{a_2}{a_1} = \ldots = \frac{a_k}{a_{k-1}} = r$$

$$\begin{aligned} \text{Thus} \quad a_1a_3 &= a_2^2, \ldots, a_{k-1}a_{k+1} = a_k^2 \\ k &= 1, 2, \ldots, n, \quad \text{or } k = 1, 2, 3, \ldots, \text{as the case may be.} \end{aligned}$$

Rewriting, we get

$$a_2 = \sqrt{a_1a_3}, a_3 = \sqrt{a_2a_4}, \ldots, a_k = \sqrt{a_{k-1}a_{k+1}},$$

a_2 is called the geometric mean (G.M.) of a_1 and a_3; a_3 that of a_2 and a_4, and so on. Generally, a_k is the G.M. of a_{k-1} and a_{k+1} If the positive real numbers $a_1, a_2, \ldots, a_n$ are in a finite geometric progression, $a_2, a_3, \ldots, a_{n-1}$ are said to be $(n-2)$ *geometric means* between the numbers a_1 and a_n. Given any two positive real numbers a, b, we can find n positive real numbers $a_2, \ldots, a_{n+1}$ such that $a = a_1, a_2, \ldots, a_{n+1}, a_{n+2} = b$ form a G.P. This procedure is known as insertion of n geometric means between any two positive numbers. Assuming that n geometric means $a_2, \ldots, a_{n+1}$ have been inserted between a and b, it then follows that b is the $(n+2)$th term of the above G.P. If r is the common ratio, then by (11.9)

$$b = a \cdot r^{n+1}$$

$$\text{So} \quad r^{n+1} = \frac{b}{a} \text{ or } r = \sqrt[n+1]{\frac{b}{a}} \tag{11.11}$$

$$\left(\text{denoted by } \left(\frac{b}{a}\right)^{1/n+1} \right)$$

Note that there exists a unique positive real number α such that $\alpha^{n+1} = \frac{b}{a}$. Then the n geometric means inserted between a and b are

$$a\alpha, a\alpha^2, \ldots, a\alpha^n$$

Remark

Our discussion of A.P. and G.P. goes through even when the terms are complex numbers except that we have difficulties in inserting geometric means between two complex numbers since square root, cube root, ..., nth root are not unique for complex numbers. This difficulty surfaces even in the case of $(n+1)$th roots when $n+1$ is even and when one of a, b is a negative real number.

Examples of G.P's are

$$1, 2, 4, 8, \ldots, 2^n, \ldots$$

for which the first term $a = 1$, and the common ratio $r = 2$, and

$$1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \ldots, (-1)^n \frac{1}{2^n}, \ldots$$

for which $a = 1$ and $r = -\frac{1}{2}$.

Example 11.8

Determine the number of terms in a geometric progression $\{a_n\}$, if $a_1 = 3, a_n = 96$ and $S_n = 189$.

Solution

If r is the common ratio, then

$$a_n = a_1 r^{n-1} = 3 \cdot r^{n-1} \qquad \text{(A)} \qquad (11.12)$$

i.e. $$3 \cdot r^{n-1} = 96 \quad \text{or} \quad r^{n-1} = 32 \qquad (11.13)$$

Now $$S_n = \frac{a_1(r^n - 1)}{r - 1} = \frac{3(r^n - 1)}{r - 1} = 189 \qquad \text{(B)} \qquad (11.14)$$

so that using (A) and (B), we get $r^n = 63r - 62$

or, $$32r = 63r - 62 \qquad (11.15)$$

Hence, $$31r = 62, \quad \text{i.e.} \quad r = 2 \qquad (11.16)$$

$$a_n = 96 = 3.2^5 = 3 \cdot 2^{n-1} \qquad (11.17)$$

so that $$n - 1 = 5 \quad \text{or} \quad n = 6 \qquad (11.18)$$

Example 11.9

Insert three geometric means between the numbers 1 and 256.

Solution

Let the numbers inserted be a_2, a_3, a_4 so that $1, a_2, a_3, a_4, 256$ are in G.P. Then assuming that the common ratio is r, the 5th term of the G.P. is

$$256 = 1 \cdot r^{5-1} = r^4$$
$$\text{or, } r^2 = 16$$
$$\text{or, } r = \pm 4$$

$$\text{If } r = 4, \text{ then } a_2 = 4,\ a_3 = 16 \text{ and } a_4 = 64$$
$$\text{If } r = -4, \text{ then } a_2 = -4,\ a_3 = 16 \text{ and } a_4 = -64$$

But geometric means are defined only for G.P's of positive numbers.

Thus $a_2 = 4$, $a_3 = 16$ and $a_4 = 64$ are the three geometric means.

Example 11.10

Find all sequences which are simultaneously arithmetic and geometric progressions.

Solution

Let $a_1, a_2, \ldots, a_n, \ldots$ be one such sequence. Then, since it is an A.P.

$$a_{n+1} = \frac{a_n + a_{n+2}}{2},\ n \geq 1.$$

Since it is a G.P, $a_1 \neq 0$, and $a_n = a_1 r^{n-1}$, where r is the common ratio. Hence

$$a_1 r^n = a_{n+1} = \frac{a_1 r^{n-1} + a_1 r^{n+1}}{2}$$

so that

$$r = \frac{1 + r^2}{2}$$

In other words $r^2 - 2r + 1 = 0$ or $(r-1)^2 = 0$ i.e. $r = 1$. This means that only a constant sequence $a, a, a, \ldots$ is both an A.P. and G.P.

Example 11.11

The sum of three numbers which are consecutive terms of A.P is 21. If the second number is reduced by 1 while the third is increased by 1, three consecutive terms of a G.P. result. Find these numbers

Solution

Let $a, a+d, a+2d$ be the three consecutive terms of A.P. The sum of these numbers $= a + a + d + a + 2d = 3(a+d)$

It is given that this sum is 21. Hence, $3(a+d) = 21$

or, $$a + d = 7 \qquad \text{(C)}$$

It is, further, given that

$$a, (a+d-1)\ ,\quad (a+2d+1)$$

form a G.P.

$$
\begin{aligned}
\text{i.e.}\qquad (a+d-1)^2 &= a(a+2d+1)\\
\text{i.e.}\qquad a^2+d^2+1+2ad-2d-2a &= a^2+2ad+a\\
\text{i.e.}\qquad (d-1)^2 &= 3a = 3\cdot(7-d) \qquad [\text{from } (C)]\\
\text{i.e.}\qquad d^2+d-20 &= 0\\
\text{i.e.}\qquad (d+5)(d-4) &= 0\\
\text{i.e.}\qquad d &= -5 \quad \text{or} \quad 4
\end{aligned}
$$

From (C) we have then, correspondingly,

$$a = 12 \quad \text{or} \quad 3$$

Thus, we have two sequences having the desired property, viz.

$$12, 7, 2$$
$$3, 7, 11$$

EXERCISE 11.3

1. Find four numbers forming a geometric progression in which the third term is greater than the first by 9, and the second term is greater than the fourth by 18.
2. The first term of a G.P. is 1. The sum of the third and fifth terms is 90. Find the common ratio of the G.P.
3. Find a G.P. for which sum of the first two terms is -4 and the fifth term is 4 times the third term.
4. If the sum of three numbers in G.P. is 38 and their product is 1728, find them.
5. If the 4th, 10th and 16th terms of a G.P. are x, y, z respectively, prove that x, y, z are in G.P. Generalise.
6. If the pth, qth and rth terms of a G.P. are a, b, c respectively, prove that
7. If a, b, c, d are in G.P., show that

$$(a^2+b^2+c^2)(b^2+c^2+d^2) = (ab+bc+cd)^2.$$

8. The sum of three numbers in A.P. is 15. If 1, 4 and 19 are added to the numbers, the resulting numbers are in G.P. Find the numbers.

9. If the first and the nth terms of a G.P. are a and b respectively and if P is the product of the first n terms, prove that

$$P^2 = (ab)^n.$$

10. If a, b, c are in G.P., prove that the following are also in G.P.:

(i) a^2, b^2, c^2 (ii) a^3, b^3, c^3

(iii) $a^2 + b^2, ab + bc, b^2 + c^2$.

11.5 Sum to infinity of a G.P.

It is of interest to see how the sum to n terms of a geometric progression

$$a, ar, ar^2, \ldots, ar^n, \ldots$$

behaves for large n, a being fixed. It is clear that the behaviour depends on the behaviour of the sequence $\{r^n\}_{n=0}^{\infty}$. Irrespective of whether r is positive or negative, it is clear that if r is numerically (i.e. in absolute value) greater than 1, then r^n becomes larger and larger in absolute value with n. Hence,

$$S_n = \frac{a(r^n - 1)}{r - 1} = \left(\frac{a}{r-1}\right)(r^n - 1)$$

also becomes larger and larger in absolute value as n increases. On the other hand, if r is,in absolute value,less than 1, it is clear that the absolute value of r^n becomes smaller and smaller as n increases. As an illustration, if $r = \frac{1}{2}$, the sequence $\{r^n\}$ is

$$1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots.$$

If $r = -\frac{1}{4}$, the sequence $\{r^n\}$ is

$$1, -\frac{1}{4}, \frac{1}{4^2}, -\frac{1}{4^3}, \ldots.$$

In both these cases, it is clear that the numerical value of r^n becomes smaller and smaller as n becomes larger and larger. In fact, given any positive number ϵ one can find a natural number n_0 such that $|r^{n_0}|, |r^{n_0+1}|, |r^{n_0+2}|, \ldots$ are all less than ϵ. This phenomenon is noticed, for that matter, for any r with $|r| < 1$. We describe this situation by saying that $r^n \to 0$, as $n \to \infty$ when $|r| < 1$. The arrow head $\to$ stands for "tends to". Thus when $|r| < 1$, $1 - r^n$ can be made to be as close to 1 as we like by choosing n sufficiently

large. This, in turn, means that $S_n = \frac{a}{1-r}(1-r^n)$ can be made to be near $\frac{a}{1-r}$ by making n sufficiently large. This is expressed by saying

$$S_n \longrightarrow \frac{a}{1-r} \text{ as } n \longrightarrow \infty$$

An equivalent phraseology is that the sum of the infinite series

$$a + ar + ar^2 + \dots$$

is $\frac{a}{1-r}$ provided $|r| < 1$. A crude way of expressing this fact is to say that the sum to infinity of the geometric progression

$$a, ar, ar^2, \dots$$

is $\frac{a}{1-r}$ provided $|r| < 1$. For instance,

$$1 + \frac{1}{2} + \frac{1}{2^2} + \dots = \frac{1}{1-\frac{1}{2}} = 2;$$

$$1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \dots = \frac{1}{1-\left(-\frac{1}{2}\right)} = \frac{1}{1+\frac{1}{2}} = \frac{1}{3/2} = \frac{2}{3}$$

The idea of the sum to infinity in a geometric progression is inherent in the infinite recurring decimal expansion of some real numbers. Take the simple case of 0.3333.... In terms of representation of rational numbers, formally this decimal stands for

$$\frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \dots.$$

$\frac{3}{10}, \frac{3}{10^2}, \frac{3}{10^3}, \dots$ is a G.P. with first term $\frac{3}{10}$ and common ratio $\frac{1}{10}(< 1)$. Thus the sum to infinity of this G.P. is

$$\frac{3}{10} \Big/ \left(1 - \frac{1}{10}\right) = \frac{3}{9} = \frac{1}{3}.$$

Thus the significance of the recurring decimal is precisely that you can get rational numbers closer and closer to $\frac{1}{3}$ by taking successively

$$.3, .33, .333, \dots$$

For example, if we take .333 in the place of $\frac{1}{3}$,

$$\frac{1}{3} - 0.333 < 0.0004 = \frac{4}{10000}$$

Thus the difference $\left| \frac{1}{3} - S_n \right|$, where

$$S_n = \frac{3}{10} + \frac{3}{10^2} + \ldots + \frac{3}{10^n}$$

is less than $\epsilon = \frac{4}{10000}$ where $n \geq 3$. It is also clear that for any $\epsilon > 0$, a stage n_0 for n can be found such that

$$\frac{1}{3} - S_n < \epsilon$$

for $n \geq n_0$. In other words, in the language introduced earlier

$$S_n \to \frac{1}{3} \text{ as } n \to \infty$$

Sometimes the recurrence of the decimals is not always simple, e.g. $0.\dot{2}\dot{3}23$. In terms of representation of rational numbers,this decimal stands for $\frac{23}{100} + \frac{23}{100^2} + \frac{23}{100^3} + \ldots$

$\frac{23}{100}, \frac{23}{100^2}, \frac{23}{100^3}$. . is a G.P. with first term $\frac{23}{100}$ and common ratio $\frac{1}{100}$

The sum to infinity of this G.P. is

$$\frac{23}{100} \Big/ \left(1 - \frac{1}{100}\right) = \frac{23}{99}$$

In the sense explained just above $\frac{23}{99}$ has, therefore, the recurring decimal expansion $.\dot{2}\dot{3}$. To be more explicit if we expand $\frac{23}{99}$ in decimals by dividing 23 by 99, the decimal expression will be 0.232323. . .

Example 11.12

Find the sum to infinity of the G.P.

$$-\frac{5}{4}, \frac{5}{16}, \frac{-5}{64}, \ldots$$

Solution

Here $a = -\frac{5}{4}$ and $r = -\frac{1}{4}$. Also $|r| < 1$.

Hence, the sum to infinity is $\dfrac{\frac{-5}{4}}{1 + \frac{1}{4}} = \dfrac{\frac{-5}{4}}{\frac{5}{4}} = -1.$

Example 11.13

Which is the rational number having the decimal expansion $0.3\overline{56}$?

Solution

$$0.3\overline{56} = 0.3 + \frac{56}{1000} + \frac{56}{100000} + \ldots$$

$$= 0.3 + \frac{56}{10^3} + \frac{56}{10^5} + \frac{56}{10^7} + \ldots$$

$$= 0.3 + \frac{56}{10^3} \Big/ \left(1 - \frac{1}{10^2}\right)$$

$$= 0.3 + \frac{56}{990} = \frac{3}{10} + \frac{56}{990}$$

$$353$$

$$990$$

EXERCISE 11.4

1. Verify that 10, −9, 8.1, . . . is a geometric progression Find the sum to infinity of the G.P.
2. The first term of a G.P. is 2 and the sum to infinity is 6. Find the common ratio.

11.6 Arithmetico-Geometric Sequence

The procedure we adopted to find the sum of n terms of a geometric progression can be used to get the sum to n terms of some sequences which are not geometric progressions. One such is what is called an arithmetico-geometric sequence. Recall that a typical arthmetic progression is

$$a, a + d, a + 2d, \ldots$$

A typical geometric progression is

$$a, ar, ar^2, \ldots$$

A typical arithmetico-geometric sequence is

$$a, (a + d)\, r, (a + 2d)\, r^2, \ldots \tag{11.19}$$

The nth term of this sequence is therefore

$$(a + \overline{n-1}d)\, r^{n-1}.$$

We shall obtain a formula for the sum S_n to n terms of the arithmetico-geometric sequence (11.19).

$$S_n = a + (a + d)\, r + (a + 2d)\, r^2 + \ldots + (a + \overline{n-1}d) r^{n-1}$$

or $$rS_n = ar + (a + d)\, r^2 + \ldots + (a + \overline{n-2}d)\, r^{n-1} + (a + \overline{n-1}d)\, r^n$$

Subtracting, we get

$$(1 - r)S_n = a + dr + dr^2 + \ldots + dr^{n-1} - (a + \overline{n-1}d) r^n$$

$$= a + dr\, \frac{1 - r^{n-1}}{1 - r} - (a + \overline{n-1}d) r^n$$

i.e. $$S_n = \frac{a}{1 - r} + dr\, \frac{1 - r^{n-1}}{(1 - r)^2} - \frac{(a + \overline{n-1}d) r^n}{1 - r}$$

When $|\, r \,| < 1$, as observed earlier

$$r^n, r^{n-1} \to 0 \text{ as } n \to \infty$$

It can also be shown that

$$n.r^n \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

Thus

$$S_n \longrightarrow \frac{a}{1-r} + \frac{dr}{(1-r)^2}, \text{ as } n \longrightarrow \infty$$

In other words, when $|r| < 1$ the sum to infinity of the arithmetico-geometric series is

$$\frac{a}{1-r} + \frac{dr}{(1-r)^2}$$

Example 11.14
Find the sum to infinity of the series

$$1 + 2.\frac{1}{3} + 3.\frac{1}{3^2} + 4.\frac{1}{3^3} + \dots$$

Solution
If S_n is the sum of the first n terms of the sequence

$$1, 2.\frac{1}{3}, 3.\frac{1}{3^2}, \dots, n.\frac{1}{3^{n-1}},$$

$$\begin{aligned} S_n &= 1 + 2.\tfrac{1}{3} + 3.\tfrac{1}{3^2} + \dots + n.\tfrac{1}{3^{n-1}} \\ \text{or, } \tfrac{1}{3}S_n &= \tfrac{1}{3} + 2.\tfrac{1}{3^2} + \dots + (n-1)\tfrac{1}{3^{n-1}} + n.\tfrac{1}{3^n}. \end{aligned}$$

Hence,

$$\begin{aligned} \tfrac{2}{3}S_n &= 1 + \tfrac{1}{3} + \tfrac{1}{3^2} + \dots + \tfrac{1}{3^{n-1}} - n\tfrac{1}{3^n} \\ &= \frac{1\left(1 - \frac{1}{3^n}\right)}{\frac{2}{3}} - n.\tfrac{1}{3^n}. \end{aligned}$$

Noting that $\frac{1}{3^n}, n.\frac{1}{3^n} \longrightarrow 0$ as $n \longrightarrow \infty$, we have

$$S_n \longrightarrow \frac{3}{2} \times \frac{3}{2} = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$$

EXERCISE 11.5

1. Sum to infinity the series

$$3 + 5.\frac{1}{4} + 7.\frac{1}{4^2} + \dots$$

2. If the sum to infinity of the series $3 + 5r + 7r^2 + \dots$ is $4\frac{8}{9}$, find r.
3. If the sum to infinity of the series
$3 + (3+d)\frac{1}{4} + (3+2d)\frac{1}{4^2} + \dots$ is $4\frac{8}{9}$, find d.

4. Sum to n terms, the series

$$1+\frac{2}{3}+\frac{3}{3^2}+\frac{4}{3^3}+\ldots$$

11.7 Sum to n Terms of Special Sequences

We shall now turn to finding the sum to n terms of some other special sequences. If the sequence is

$$1, 2, 3, 4, \ldots, n, n+1, \ldots$$

i.e. the sequence of natural numbers, then we have seen [vide sections 11.3, equation (11.7)], the sum S_n of n terms of the sequence is given by

$$S_n=\frac{n(n+1)}{2}, \qquad n=1,2,\ldots$$

We shall now consider the sequence of squares of the natural numbers, viz.

$$1^2, 2^2, 3^2, \ldots, n^2, (n+1)^2, \ldots$$

If S_n is the sum to n terms of this sequence i.e., if

$$S_n=1^2+2^2+\ldots+n^2=\sum_{k=1}^{n} k^2,$$

we will prove that

$$S_n=\frac{n(n+1)(2n+1)}{6}$$

Consider the identity

$$(x+1)^3-x^3=3x^2+3x+1$$

Putting $x=n,(n-1),(n-2),\ldots,1$ in this identity, we have successively

$$\begin{aligned}
(n+1)^3-n^3&=3n^2+3n+1\\
n^3-(n-1)^3&=3(n-1)^2+3(n-1)+1\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
2^3-1^3&=3\cdot1^2+3\cdot1+1
\end{aligned}$$

Adding these equations we have

$$(n+1)^3-1^3=3\sum_{k=1}^{n} k^2+3\sum_{k=1}^{n} k+n$$

i.e. $n^3 + 3n^2 + 3n = 3\sum_{k=1}^{n} k^2 + \frac{3n(n+1)}{2} + n$

$$\begin{aligned}
\text{Hence, } S_n = \sum_{k=1}^{n} k^2 = \frac{n^3+3n^2+3n}{3} &- \frac{n(n+1)}{2} - \frac{n}{3} \\
&= \frac{2n^3+6n^2+6n-3n^2-3n-2n}{6} \\
&= \frac{n(2n^2+3n+1)}{6} \\
&= \frac{n(n+1)(2n+1)}{6}
\end{aligned}$$

We shall next get an expression for the sum of the cubes of the first n natural numbers, viz.

$$S_n = \sum_{k=1}^{n} k^3$$

Here we use the identity

$$(x+1)^4 - x^4 = 4x^3 + 6x^2 + 4x + 1$$

As earlier, we put $x = n, (n-1), (n-2), \ldots, 1$ in the above identity and add both sides of the resulting n equations.

We, then, have

$$\begin{aligned}
(n+1)^4 - 1^4 &= 4\sum_{k=1}^{n} k^3 + 6\sum_{k=1}^{n} k^2 + 4\sum_{k=1}^{n} k + n \\
&= 4S_n + 6\frac{n(n+1)(2n+1)}{6} + 4\frac{n(n+1)}{2} + n \\
&= 4S_n + n(n+1)(2n+1) + 2n(n+1) + n
\end{aligned}$$

So

$$\begin{aligned}
S_n &= \frac{1}{4}\left[n^4 + 4n^3 + 6n^2 + 4n - n(n+1)(2n+1) - 2n(n+1) - n\right] \\
&= \tfrac{1}{4}\left[n^4 + 2n^3 + n^2\right] \\
&= \tfrac{1}{4}n^2(n^2 + 2n + 1) \\
&= \tfrac{1}{4}n^2(n+1)^2 = \left\{\frac{n(n+1)}{2}\right\}^2
\end{aligned}$$

Remark

We note that the sum of the cubes of the first n natural numbers is the square of the sum of the first n natural numbers.

Example 11.15

Find $1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2$

Solution

$1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2$
$= 1^2 + 2^2 + 3^2 + \ldots + (2n)^2 - \left[2^2 + 4^2 + \ldots + (2n)^2\right]$

$= S_{2n} - 4S_n$ (Where S_k is the sum of the squares of the first k natural numbers).

$$= \frac{1}{6}\cdot 2n(2n+1)(4n+1) - \frac{4}{6}n(n+1)(2n+1)$$
$$= \frac{1}{3}[n(2n+1)\{4n+1-2(n+1)\}]$$
$$= \frac{1}{3}n(2n+1)(2n-1).$$

Example 11.16

Sum to n terms the series whose nth term is $n(n+1)(n+4)$

Solution

The nth term is:

$$n(n+1)(n+4) = n(n^2+5n+4)$$
$$= n^3+5n^2+4n$$

Thus, the sum to n terms S_n is given by

$$S_n = \sum^{n} k^3 + 5\sum^{n} k^2 + 4\sum_{k=1}^{n} k$$

$$= \left[\frac{n(n+1)}{2}\right]^2 + \frac{5}{6}n(n+1)(2n+1) + 4\cdot\frac{n(n+1)}{2}$$

$$12 \quad \left[3n^2+23n+34\right]$$

Example 11.17

The sequence N of natural numbers is divided into classes as follows:

$$\begin{array}{cccccc} & & 1 & 2 & & \\ & 3 & 4 & 5 & 6 & \\ 7 & 8 & 9 & 10 & 11 & 12 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

Show that the sum of the numbers in the nth row is $n(2n^2+1)$.

Solution

The number of numbers in the nth row is clearly $2n$. The number of numbers up to the nth row (including nth row) is $2+4+\ldots+2n = 2(1+2+\ldots+n) = n(n+1)$. If S_n is the sum of the first n natural numbers, then the desired sum is

$$S_{n(n+1)} - S_{(n-1)n} = \frac{n(n+1)\{n(n+1)+1\}}{2} - \frac{(n-1)n\{(n-1)n+1\}}{2}$$
$$= \frac{n(n+1)(n^2+n+1)-(n-1)n(n^2-n+1)}{2}$$
$$= \frac{n}{2}\{n^3+2n^2+2n+1-n^3+2n^2-2n+1\}$$
$$= \frac{n}{2}(4n^2+2) = n(2n^2+1).$$

Example 11.18

An odd number of stones lie along a straight path, the distance between consecutive stones being 10 m. The stones are to be collected at the place where the middle stone lies. A man can carry only one stone at a time. He starts carrying the stones beginning from the extreme stone. If he covers a path of 3 km, how many stones are there?

Solution

Assume that there are $n = 2k + 1$ stones so that there are k stones to the right and k stones to the left of the middle stone. The distance covered by the man to bring to the middle position the extreme right stone is $10\,k$ metres. To bring the stone which lies in the $(k-1)$th place on the right the distance covered is $2 \times 10(k-1)$. Proceeding thus to collect all the stones on the right of the middle position, the man has to cover in metres a distance of

$$S_1 = 10k + 20(k-1) + 20(k-2) + \ldots + 20\cdot 1$$

Note that this distance is independent of the order in which the stones are collected. To collect now the stones on the left, the man has to cover a distance of $10k$ metres to reach the left extreme stone and thereafter do as he did for the stones on the right. In other words, the distance covered by him to collect the stones on the left would be

$$10k + S_1$$

Now, the total distance covered is

$$\begin{aligned} S_1 + S_1 + 10k &= 30k + 40[(k-1) + (k-2) + \ldots + 1] \\ &= 3000 \text{ (given)} \end{aligned}$$

$$\begin{aligned} \text{i.e. } 3000 &= 30k + 40\frac{(k-1)k}{2} \\ &= 30k + 20k(k-1) \\ &= 10k + 20k^2 \end{aligned}$$

i.e. $2k^2 + k - 300 = 0$

i.e. $2k^2 - 24k + 25k - 300 = 0$

i.e. $2k(k-12) + 25(k-12) = 0$

i.e. $(k-12)(2k+25) = 0$

Hence, $k = 12$ or $k = -\frac{25}{2}$.

The latter value being inadmissible, $k = 12$ or $n = 2k + 1 = 24 + 1 = 25$. The number of stones is, therefore, 25.

Example 11.19

Prove, by mathematical induction, that

$$1^2 + 2^2 + \ldots + n^2 = \frac{1}{6}n(n+1)(2n+1)$$

Solution

For $n = 1$, the left side $= 1^2 = 1$, The right side $= \frac{1}{6}1\cdot2\cdot3 = 1$.
Hence the result is true for $n = 1$. Assume that the result is true for $n = k$. i.e. assume that,

$$1^2 + 2^2 + 3^2 + \ldots + k^2 = \frac{1}{6}k(k+1)(2k+1) .$$

Now $1^2 + 2^2 + \ldots + k^2 + (k+1)^2$

$$\begin{aligned}
&= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2 \\
&= \frac{1}{6}(k+1)\{k(2k+1) + 6(k+1)\} \\
&= \frac{1}{6}(k+1)(2k^2 + 7k + 6) \\
&= \frac{1}{6}(k+1)(k+2)(2k+3) \\
&= \frac{1}{6}(k+1)\{(k+1)+1\}\{2(k+1)+1\},
\end{aligned}$$

which shows that the result is true for $n = k + 1$. Thus by the principle of mathematical induction the result is true for any $k = 1, 2, \ldots$.

EXERCISE 11.6

1. Find $2^2 + 4^2 + 6^2 + \ldots + (2n)^2$.
2. Find $1^3 + 3^3 + 5^3 + \ldots + (2n-1)^3$.
3. Use mathematical induction to prove that

$$1^3 + 2^3 + \ldots + n^3 = \frac{1}{4}n^2(n+1)^2.$$

MISCELLANEOUS EXERCISE ON CHAPTER 11

1. If the pth, qth, rth terms of an A.P. are x, y, z respectively, show that

$$x(q-r) + y(r-p) + z(p-q) = 0.$$

2. If the sum of the first n terms of a sequence is of the form $An^2 + Bn$ where A, B are constants independent of n, show that the sequence is an A.P. Is the converse always true? Justify your answer.
3. There are n arithmetic means between 3 and 17. The ratio of the last mean to the first mean is 3:1. Find the value of n.
4. The sum of three numbers in A.P. is -3, and their product is 8. Find the numbers.
5. The digits of a positive integer, having three digits, are in A.P. and their sum is 15. The number obtained by reversing the digits is 594 less than the original number. Find the number.
6. Two cars start together in the same direction from the same place. The first goes with uniform speed of 10 km/h. The second goes at a speed of 8 km/h in the first hour and increases the speed by 1/2 km each succeeding hour. After how many hours will the second car overtake the first if both cars go non-stop?

7. An insect starts from a point and travels in a straight path one mm in the first second and half of the distance covered in the previous second in the succeeding second. In how much time would it reach a point 3 mm away from its starting point?

8. The first term of a geometric progression is 1. The sum of the third and fifth terms is 90. Find the common ratio of the geometric progression.

9. The sum of three numbers in geometric progression is 56. If we subtract 1, 7, 21 from these numbers in that order, we obtain an arithmetic progression. Find the numbers.

10. The inventor of the chess board suggested a reward of one grain of wheat for the first square, 2 grains for the second, 4 grains for the third and so on, doubling the number of the grains for subsequent squares. How many grains would have to be given to the inventor? (There are 64 squares in the chess board).

11. If a, b are two numbers, by their harmonic mean c is meant such that $\frac{1}{a}, \frac{1}{c}, \frac{1}{b}$ are in A.P. If a, b are positive and A, G, H denote respectively the arithmetic, geometric and harmonic means of a, b, show that A, G, H form a G.P.

12. If S_1, S_2, S_3 are the sums of the first n natural numbers, their squares and their cubes respectively, show that
$$9S_2^2 = S_3(1 + 8S_1).$$

13. Show that
$$\frac{1 \times 2^2 + 2 \times 3^2 + \ldots + n \times (n+1)^2}{1^2 \times 2 + 2^2 \times 3 + \ldots + n^2 \times (n+1)} =$$

14. Show that the sum of the cubes of any number of consecutive integers is divisible by the sum of those integers.

15. Find the sum to infinity of the series
$$1 + \frac{3}{2} + \frac{5}{2^2} + \frac{7}{2^3} + \ldots$$

16. If m times the mth term of an A.P. is equal to n times its nth term, then show that $(m+n)^{th}$ term of the A.P. is zero.

17. Determine the 25th term of the A.P., whose 9th term is -6 and common difference is $5/4$.

18. If $\frac{2}{3}$, k, $\frac{5}{8}$, are in A.P, find the value of k.

19. If pth term of an A.P. is q, and the qth term is p, show that their rth term is $p+q-r$.

20. How many terms of the sequence $-12, -9, -6, -3, \ldots$ must be taken to make the sum 54?

21. Find four numbers in A.P. whose sum is 20 and the sum of whose squares is 120.

22. Find the sum of 50 terms of the sequence $7, 7.7, 7.77, 7.777, \ldots$

23. How many terms of the sequence $3, 3^2, 3^3, \ldots$ are needed to give the sum 120?

24. Find the sum of the following series:
    (i) $5 + 55 + 555 + \ldots$ n terms
    (ii) $.6 + .66 + .666 + \ldots$ n terms

25. If the first term of a G.P. is 729 and the 7th term is 64, find S_∞.

26. If the A.M. between two positive numbers is 34 and their G.M. is 16, find the numbers.

27. Find the sum of the series $1 - \frac{3}{2} + \frac{5}{4} - \frac{7}{8} + \ldots$

28. Find the sum of n terms of
    (i) $1\cdot2\cdot3 + 2\cdot3\cdot4 + 3\cdot4\cdot5 + \ldots$
    (ii) $1\cdot2^2 + 2\cdot3^2 + 3\cdot4^2 + \ldots$

CHAPTER 12

Permutations and Combinations

12.1 Introduction

In some counting problems, we can find out the answer without actually counting fully. For example, consider the problem : How many trees are there in the diagram below?

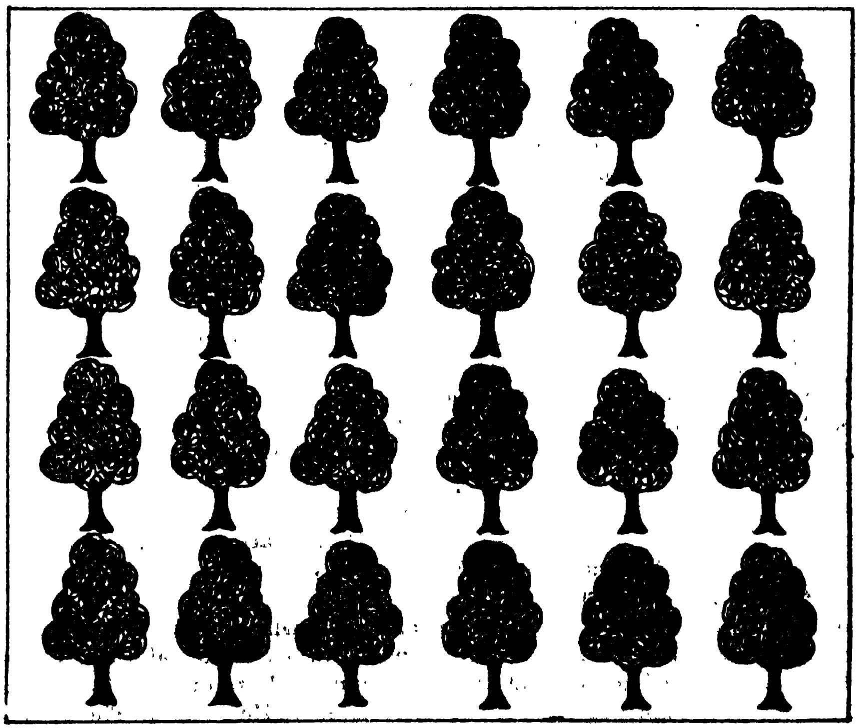


Fig 12.1

We need not count them from the first to the last. It is enough to observe that there are four rows and in each row there are six trees. So, we conclude even without fully counting that there are twenty four trees. Here, multiplication is the idea that has simplified the counting.

In this chapter, we shall see some other techniques of answering some counting problems without actually counting, or even listing, all of them.

Such problems have been of interest since thousands of years, as the following passage from Mahabharata indicates: "O King: I am surprised to find (after a long verification) that your quick counting has indeed given the correct answer. Will you please teach me the techniques therein?," Bahuka asked the king Rituparna. Thereafter the king taught him the art of counting fast.

12.2 Fundamental Principle of Counting

We start with an example from day-to-day life. Ram was allotted a roll number for his examination. But he forgot his number. What all he remembered was that it was an even two-digit number without zero. How many such numbers are there?

One way to count them is to list all of them as under:

12	22	32	42	52	62	72	82	92
14	24	34	44	54	64	74	84	94
16	26	36	46	56	66	76	86	96
18	28	38	48	58	68	78	88	98

and then count them as $9 \times 4 = 36$

Is there a cleverer method to arrive at this answer 36 without listing them at all?

Let us see.

The digit in the unit's place can be any one of the four digits 2, 4, 6, 8.

Explanation: This is because our number is an even number without involving zero.
The digit in the ten's place can be any one of the nine digits 1, 2, 3, 4, 5, 6, 7, 8, 9.

Thus there are nine ways to fill up one of the two digits, and there are four ways to fill up the other. Totally, therefore, there are $9 \times 4 = 36$ ways to fill up the two digits as required.

This example illustrates the following general principle:

Fundamental Principle of Counting: If an event can happen in exactly m ways, and if following it, a second event can happen in exactly n ways, then the two events in succession can happen in exactly $m \times n$ i.e. mn ways.

Explanation: This is because for each of the m ways in which the first event can happen, there are n ways in which the second can happen. In the example above, the first event is the filling up of the left-digit of the two-digit number. Here $m = 9$. The second event is the filling up of the right-digit of the two-digit number. Here $n = 4$. The total number of 2-digit numbers satisfying our requirements is, therefore $9 \times 4 = 36$.

The same principle can be generalised to three or more events occurring in succession.

Example 12.1

It has been decided that the flag of a newly formed forum will be in the form □□□ of three blocks, each coloured differently. If there are six different colours on the whole to choose from, how many such designs are possible?

Solution

The first block can be coloured in 6 ways, because there are 6 colours to choose from.

The second block can be coloured only in 5 ways, because among the 6 colours, one would have been used for the first block, and there are 5 remaining colours to choose from.

Similarly, the third block can be coloured only in 4 ways. Therefore, by the fundamental principle of counting, the number of flag-designs as required is $6 \times 5 \times 4 = 120$.

Example 12.2

In a class there are 27 boys and 14 girls. The teacher wants to select 1 boy and 1 girl to represent the class in a function. In how many ways can the teacher make this selection?

Solution

Here the teacher is to perform two operations: (i) selecting a boy from among the 27 boys, and (ii) selecting a girl from among the 14 girls. The first of these can be done in 27 ways (since any one of the 27 can be selected) and the second can be performed in 14 ways. Therefore, by the fundamental principle of counting, the required number of ways is $27 \times 14 = 378$.

Example 12.3

(i) How many numbers are there between 100 and 1000 such that 7 is in the unit's place?

(ii) How many numbers are there between 100 and 1000 such that at least one of their digits is 7?

(iii) How many of them have exactly one of their digits as 7?

Solution

(i) First note that all these numbers have three digits. 7 is in the unit's place. The middle digit can be any one of the 10 digits between 0 and 9. The digit in hundred's place can be any one of the 9 digits between 1 and 9. Therefore, by the fundamental principle of counting, there are $10 \times 9 = 90$ numbers between 100 and 1000, having 7 in the unit's place.

(ii) Total number of 3 digit numbers having atleast one of their digits as 7 = (Total number of three digit numbers) – (Total number of 3 digit numbers in which 7 does not appear at all)

$= (9 \times 10 \times 10) - (8 \times 9 \times 9)$

$= 900 - 648$

$= 252$

(iii) We want to count all the 3-digit numbers having 7 as exactly one of their digits. Here again there are three kinds of numbers:

First, those numbers that have 7 in the unit's place but not in any other place.

Secondly, those numbers that have 7 in the ten's place but not in any other place.

Thirdly, those numbers that have 7 in the hundred's place but not in any other place.

We shall count these three kinds of numbers separately and add them up to arrive at our answer.

In the first kind of numbers, the unit's place has 7, the ten's place can have any one of the digits except 7, and there are 9 possibilities here (namely 0, 1, 2, 3, 4, 5, 6, 8, 9); the hundred's place can have any one of the digits except 0 and 7, and there are 8 possibilities here. Thus there are $9 \times 8 = 72$ numbers of the first kind.

Similarly, in a number of the second kind, the ten's place is fixed, the hundred's place has 8 possibilities and the unit's place has 9 possibilities. Therefore, there are $8 \times 9 = 72$ such numbers.

Similarly, in a number of the third kind, the hundred's place is fixed, the other two places can be occupied by any one of the digits except 7. There are $9 \times 9 = 81$ such numbers.

Note that these three kinds are mutually exclusive and that no number has been counted in more than one of them.

The required number is $72 + 72 + 81 = 225$.

Explanation: We have shown that there are 252 numbers with 7 as at least one of their digits and that there are 225 of them with 7 exactly one of their digits. It means that there are $252 - 225 = 27$ numbers having 7 in more than one place. This can be directly checked (and used to give an alternate solution for (iii) using (ii)). These are the nine numbers 177, 277, 377, 477, 577, 677, 777, 877, 977, the nine numbers 707, 717, 727, 737, 747, 757, 767, 787, 797 and the nine numbers between 770 and 779 (except 777 that has already been counted).

Example 12.4

In how many ways can this diagram be coloured subject to the following two conditions? (i) Each of the smaller triangle is to be painted with one of three colours: red, blue or green, and (ii) no two adjacent regions receive the same colour?

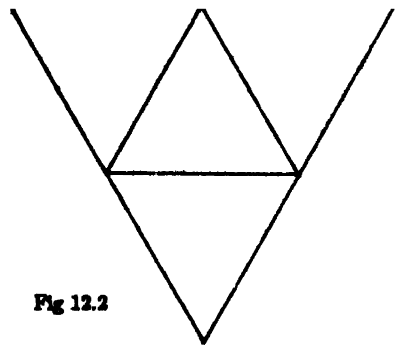


Fig 12.2

Solution

These conditions are satisfied exactly when we do as follows. First paint the central triangle in any one of the three colours. Next paint the remaining 3 triangles, with any one of the remaining two colours. By the fundamental principle of counting, this can be done in

$$3 \times 2 \times 2 \times 2 = 24 \text{ ways}$$

EXERCISE 12.1

1. A lady wants to select one cotton saree and one polyester saree from a textile shop. If there are 10 cotton varieties and 12 polyester varieties, in how many ways can she choose the two sarees?
2. In a monthly test, the teacher decides that there will be three questions, one from each of Exercises 7, 8 and 9 of the textbook. If there are 12 questions in Exercise 7, 18 in Exercise 8 and 9 in Exercise 9, in how many ways can the three questions be selected?
3. How many three-digit numbers can be formed without using the digits 0, 2, 3, 4, 5, and 6?
4. How many numbers are there between 100 and 1000 in which all the digits are distinct?
5. How many words (with or without meaning) of three distinct letters of the English alphabets are there?
6. The students in a class are seated according to their marks in the previous examination. Once it so happens that four of these students get equal marks and therefore the same rank. To decide their seating arrangement, the teacher wants to write down all possible arrangements, one in each of separate bits of paper, in order to choose one of these by lots. How many bits of paper are required?
7. From Madras to Hyderabad, there are three routes; air, rail and road. From Hyderabad to Vikarabad, there are two routes, rail and road. From Madras to Vikarabad via Hyderabad, how many kinds of routes are there?

8. A mint prepares metallic calendars specifying months, dates and days in the form of monthly sheets (one plate for each month). How many types of February calendars should it prepare to serve for all the possibilities in the future years?
9. For a set of five true-or-false questions, no student has written the all-correct answers, and no two students have given the same sequence of answers. What is the maximum number of students in the class, for this to be possible?
10. How many numbers are there between 100 and 1000 such that every digit is either 2 or 9?
11. Each section in the first year of plus two course has exactly 40 students. If there are 4 sections, in how many ways can a set of 4 student-representatives be selected, one from each section?

12.3 The Factorial Introduction

In this short section, we introduce the term and notation of the factorial. This will be often used in all the sections that follow in this chapter.

Definition and Notation

Consider the products
1
1·2
1·2·3
1·2·3·4
etc.
We denote them respectively by $1!, 2!, 3!, 4!$ etc. In general, $n!$ denotes the product of the first n natural numbers.

$$n! = 1\cdot 2\cdot 3\ldots \cdot n$$

This $n!$ is read as "n factorial". Thus, for example

$$4! = 24 \text{ because } 1\cdot 2\cdot 3\cdot 4 = 24$$

$$5! = 120 \text{ because } 1\cdot 2\cdot 3\cdot 4\cdot 5 = 120$$

Alternate Notation: The notation $\underline{|n}$ is also used sometimes in place of $n!$
The zero factorial: What is 0!? It does not make sense to define it as the product of integers from 1 up to 0. We do not leave it undefined, because we require it in the later sections.

We define 0! to be equal to 1. By defining 0! to be 1, we make sure that many of the properties of $n!$ remain true even when $n = 0$. For instance, for all $n \geq 1$, it can be easily proved that

$$(n+1)! = (n+1) \times n!$$

If we take $0! = 1$, then this result will be true even for $n = 0$ (as you can see).

Remark

We do not define the factorial of proper fractions or negative integers. The factorial defined only for whole numbers.

Example 12.5

Prove that

$$n!(n+2) = n! + (n+1)!$$

Solution

$$\begin{aligned}\text{The right hand side } &= n! + (n+1)! \\ &= n! + n!(n+1) \\ &= n![1 + (n+1)] \\ &= n!(n+2) \\ &= \text{The left hand side}\end{aligned}$$

Explanation: Here we have used: $(n+1)! = n!(n+1)$

This is because $\begin{aligned}(n+1)! &= 1\cdot2\cdot3\ldots..n.(n+1) \\ &= n!(n+1)\end{aligned}$

Example 12.6

If $(n+2)! = 2550(n!)$, find n.

Solution

$$\begin{aligned}(n+2)! &= 1\cdot2\cdot3\ldots.\ n(n+1)(n+2) \\ &= n!(n+1)(n+2)\end{aligned}$$

This is given to be equal to 2550 $(n!)$

Therefore, we get $(n+1)(n+2) = 2550$

$$\text{or,}\quad n^2 + 3n - 2548 = 0$$

This is a quadratic equation in n. Its roots are

$$n = \frac{-3 \pm \sqrt{9 + 4 \times 2548}}{}$$

$$-3 \pm 101$$

$$= 49 \text{ or } -52.$$

Of these two values of n, the second is ruled out because, we do not speak of $n!$ when n is negative. Therefore $n = 49$.

Explanation: Note the importance of brackets in expressions like $(n+2)!$, $2550(n!)$, etc. When the brackets are removed, $n+2!$ means $n+(1\times 2)$. $2550n!$ gives room for a doubt that the factorial may be taken for $2550n$.

Example 12.7
Find the value of $\frac{10!}{5!5!}$.

Solution

$$\frac{10!}{5!5!} = \frac{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9\cdot10}{1\cdot2\cdot3\cdot4\cdot5\cdot1\cdot2\cdot3\cdot4\cdot5}$$
$$= \frac{7\cdot8\cdot9\cdot10}{4\cdot5} = 7\cdot4\cdot9 = 252$$

Remark

Expressions of the form $\frac{10!}{5!5!}$, that is, of the form $\frac{n!}{r!(n-r)!}$ will be frequently met in later sections.

EXERCISE 12.2

1. Compute $\frac{30!}{28!}$.
2. Which of the following are true?
   $5(4!) = (5\times 4)!$
   $(2+3)! = 2!+3!$
   $(2\times 3)! = 2!\times 3!$
3. If $\frac{1}{9!}+\frac{1}{10!}=\frac{x}{11!}$, find x.
4. Find the LCM of the numbers $4!$, $5!$ and $6!$.
5. If $(n+1)! = 12[(n-1)!]$, find n.
6. When $n=5$ and $r=2$, find the values of $\frac{n!}{r!}$ and $\frac{n!}{r!(n-r)!}$.
7. If $\frac{n!}{2!(n-2)!}$ and $\frac{n!}{4!(n-4)!}$ are in the ratio 2:1, find the value of n.
8. Prove that $\frac{(2n)!}{n!} = 1\cdot3\cdot5\ldots(2n-1)2^n$.

9. Prove the inequalities $(n!)^2 \leq n^n \cdot n! < (2n)!$ for all positive integers n.

10. Convert into factorials: (i) $4\cdot5\cdot6\cdot7\cdot8\cdot9\cdot10\cdot11$, (ii) $2\cdot4\cdot6\cdot8\cdot10$

11. If $(n+2)! = 20(n!)$, find n.

12. Prove that there are exactly $4!$ numbers between 1000 and 10000 that contain the digits 1,3,5 and 7.

13. Prove that $33!$ is divisible by 2^{15}. What is the largest integer n such that $33!$ is divisible by 2^n?

14. Prove that $n! + 1$ is not divisible by any number between 2 and n.

12.4 Permutations

The word permutation means 'any of the ways in which a set of objects can be arranged'. For example, consider the set { pen, chair, teacher}. Then "chair, pen, teacher" is a permutation of this set. "pen, chair, teacher" is another permutation. "teacher, pen, chair" is one more permutation of this set. Note that the order of arrangement is taken into account; when the order is changed, a different permutation results.

Example 12.8

Write down all the permutations of the set of three letters A, B, C.

Solution

The desired permutations are:

A B C
A C B
B C A
B A C
C A B
C B A

i.e., there are six permutations.

Example 12.9

Write down all the permutations of the English vowels A, E, I, O, U taken three at a time, and starting with A.

Solution

The desired permutations are:

A	E	I
A	E	O
A	E	U
A	I	O
A	I	U
A	O	U
A	I	E
A	O	E
A	U	E
A	O	I
A	U	I
A	U	O

i.e., there are 12 such permutations.

A Notation: In example 12.8, we find that there are six permutations, on a set of three letters, taken all at a time. We denote this fact by

$$P(3,3) = 6$$

In example 12.9, we find that there are 12 permutations, on a set of five letters, taken three at a time, starting with A. There will be 12 such permutations starting with E, and so on. Totally, there are 60 permutations on a set of 5 letters, taken 3 at a time. We denote this fact by

$$P(5,3) = 60$$

More generally, if n and r are integers such that $1 \leq r \leq n$, the symbol $P(n,r)$ denotes the number of permutations of n elements, taken r at a time.

Theorem 12.1

Let $1 \leq r \leq n$. Then

$$P(n,r) = n(n-1)(n-2)\dots(n-r+1)$$

Explanation: When $r = 1$, the expression in the right side is interpreted as n. When $r = 2$, it is interpreted as $n(n-1)$. This is because it means the product of all integers between n and $(n-r+1)$.

Proof of the Theorem: While arranging n objects taken r at a time, the first position can be filled by any one of the n objects. This can be done in n ways. The second position can be filled by any one of the remaining $n-1$ objects. This can be done in $n-1$ ways and so on. Finally, the rth position can be filled in $n-r+1$ ways, because after filling up the previous $r-1$ positions, exactly $n-(r-1) = n-r+1$ objects remain. After filling up the rth position, we are through in arranging r objects.

By the fundamental principle of counting, the total number of ways of arranging r objects from n objects is $n(n-1)\dots(n-r+1)$. Thus

$$P(n,r) = n(n-1)\dots(n-r+1)$$

Restatement: Let $1 \leq r \leq n$. Then

$$P(n,r) = \frac{n!}{(n-r)!}$$

Proof: Noting that $n! = 1{\cdot}2 \ldots .(n-1){\cdot}n$
and $(n-r)! = 1{\cdot}2 \ldots .(n-r-1)(n-r)$,
we observe that many factors are common to both the numerator and the denominator of $\frac{n!}{(n-r)!}$. After cancelling them, $\frac{n!}{(n-r)!} = (n-r+1)(n-r+2)\ldots n$.
This is the same expression for $P(n,r)$ obtained in the above Theorem.

Explanation: When $r = n$, $(n-r)! = 0! = 1$ and therefore $P(n,n) = n!$. This can also be derived directly from the theorem.
Corollary: $P(n,n) = n!$

Remark

The formula $P(n,n) = n!$ is easily verified in some particular cases. We have already seen that $P(3,3) = 6$ and $P(5,5) = 120$, whereas we already know that $3! = 6$ and $5! = 120$.

Remark

When $r > n$, we do not talk of $P(n,r)$. Because, we cannot arrange more objects, taken from less number of objects. So also, in our formula $\frac{n!}{(n-r)!}$, there is no sense when $r > n$, because factorial of a negative number is not defined.

Formulae to Remember

When $1 \leq r \leq n$,

$$P(n,r) = n(n-1)\ldots(n-r+1) = \frac{n!}{(n-r)!}$$

$$P(n,n) = n!$$

Remark

When $r = 0$, some meaning can be attached to $P(n,r)$. In fact $P(n,0)$ should be the number of permutations of n objects taken 0 at a time. There is only one such permutation, taking nothing at all. Therefore $P(n,0) = 1$. Note that this is in conformity with the formula $P(n,r) = \frac{n!}{(n-r)!}$

Example 12.10
Find the value of $P(4,3)$.

Solution

We know $P(n,r) = n(n-1)\dots(n-r+1)$
Here $n = 4, r = 3, n-r+1 = 4-3+1 = 2$
$\therefore P(n,r) = 4{\cdot}3{\cdot}2 = 24$

Example 12.11

If $2P(5,3) = P(n,4)$, find n.

Solution

$$\text{Note that } 2 \times P(5,3) = 2 \times 5 \times 4 \times 3 = 2 \times 3 \times 4 \times 5 = 5!$$

Hence, $P(n,4) = 5!$,

i.e. $n(n-1)(n-2)(n-3) = 5 \times 4 \times 3 \times 2$

This shows (as n is a natural number) that $n = 5$. For, if $n \in N, n \geq 6$, the left hand side is greater than the right. n cannot be 0, 1, 2, 3. If $n = 4$ the left hand side is less than the right. So $n = 5$.

EXERCISE 12.3

1. Prove that $P(n,n) = 2P(n,n-2)$
2. Prove that $P(10,3) = P(9,3) + 3P(9,2)$
3. Prove that $P(n,r) = P(n-1,r) + rP(n-1,r-1)$
4. Prove that if $r \leq s \leq n$, then $P(n,s)$ is divisible by $P(n,r)$.
5. If $P(5,r) = 2P(6,r-1)$, find r.
6. If $5P(4,r) = 6P(5,r-1)$, then find r.
7. If $P(5,r) = P(6,r-1)$, prove that $r = 4$.
8. If $P(11,r) = P(12,r-1)$, find r.
9. If $P(n-1,3) : P(n,4) = 1 : 9$, find n.

12.5 Practical Problems on Permutations

Example 12.12
It is required to seat 5 men and 4 women in a row so that the women occupy the even places. How many such arrangements are possible?

Explanation: In the row of 9 positions, the second, fourth, sixth and the eighth are the even places.

Solution
There are exactly four even places, and exactly 4 women to occupy them. Therefore, these even positions can be filled in $P(4,4)$ ways (ways of arranging 4 women in 4 positions). The remaining 5 positions can be filled up by the 5 men in $P(5,5)$ ways. Therefore,by the fundamental principle of counting, the number of seating arrangements as required, is $P(4,4)\cdot P(5,5) = 4!.5! = 24 \times 120 = 2880$.

Example 12.13
How many three-digit numbers are there, with no digit repeated?

Solution
There are as many such numbers as there are permutations of the ten digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, taken three at a time, with the condition that 0 is not in the left-most position. This is $P(10,3) - P(9,2)$ because, $P(10,3)$ is the number of all permutations of the ten digits taken three at a time, and $P(9,2)$ is the number of such arrangements with 0 in the left-most position and therefore disallowed.

Explanation: When 0 is in the left-most position, the other 9 digits are to be arranged in the other two positions.

$P(10,3) - P(9,2) = 10 \times 9 \times 8 - 9 \times 8 = 648$

Explanation: You may ask, 'Why don't we do this problem directly by the fundamental principle of counting?' The hundred's place can be filled by any one of the nine non-zero digits from 1 to 9. The ten's place can be any one of the nine digits, except the one that is in the hundred's place. The unit's place can be any one of the remaining 8 digits. Therefore, the answer is $9 \times 9 \times 8$. Do you see that we get the same answer?

It is interesting to note that a single problem can be done in many ways, arriving at the same answer. Here is one more method: The three-digit numbers without repeated digits are of three kinds: First, those that do not contain zero. There are $P(9,3)$ numbers of this kind. Next, those that contain 0 as the middle digit. Since the other two places are to be filled by the other 9 digits, there are $P(9,2)$ numbers of this kind. Lastly, those that contain 0 in the unit's place, and there are $P(9,2)$ numbers of this kind.
Therefore, there are $P(9,3) + P(9,2) + P(9,2)$ numbers. Do you see that this also gives the same answer 648?

Example 12.14

There are 6 items in column A and 6 items in column B. A student is asked to match each item in column A with an item in column B. How many possible (correct or incorrect) answers are there to this question?

Solution

Each answer to this question is an arrangement of the 6 items of column B so as to correspond to the given arrangement of items in column A. Therefore, there are as many answers as these are permutations of 6 objects. Hence, our answer is $P(6,6) = 6! = 720$.

EXERCISE 12.4

1. There are 3 different rings to be worn in four fingers with at most one in each finger. In how many ways can this be done?
2. In how many ways can five children stand in a queue?
3. Four books, one each in Chemistry, Physics, Biology and Mathematics, are to be arranged in a shelf. In how many ways can this be done?
4. Ten students are participating in a race. In how many ways can the first three prizes be won?
5. Four letters E, K, S and V, one in each, were purchased from a plastic warehouse. How many ordered pairs of letters, to be used as initials, can be formed from them?
6. How many words, with or without meanings, can be formed using all the letters of the word EQUATION, using each letter exactly once?
7. How many four-digit numbers are there with distinct digits?
8. In an examination hall, there are four rows of chairs. Each row has 8 chairs one behind the other. There are two classes sitting for the examination, with 16 students in each class. It is desired that in each row, all students belong to the same class and that no two adjacent rows are allotted to the same class. In how many ways can these 32 students be seated?
9. How many three-digit numbers are there, with distinct digits, with each digit odd?
10. From among the 36 teachers in a school, one principal and one vice-principal are to be appointed. In how many ways can this be done?

12.6 Permutations under Certain Conditions

In actual life, the problems of counting the number of arrangements do not always occur in the simplest form. Sometimes, repetitions are allowed in the arrangements. Sometimes, distinctions between some of the objects are ignored. Sometimes, certain distinct arrangements are considered to be the same for other reasons. Let us see some examples.

Example 12.15
A child has plastic toys bearing the digits 2,2 and 5. How many three-digit numbers can he make, using them?

Solution
Let us list all the possible numbers that can be formed using all the three toys. They are 225, 252 and 522. Therefore, the answer is three.
Explanation: Even though there are 3 toys to be arranged, the number of arrangements here is less than $P(3,3)$. This is because two of these three toys are considered indistinguishable.

Example 12.16
There are five round stickers. 3 of them are red and the other 2 green. It is desired to make a design by pasting them in a row. How many such designs are possible?
Explanation: There are $P(5,5)$ arrangements of these 5 stickers. But since the stickers are distinguishable only by their colour, many arrangements may give the same design. How many permutations corresponds to one design? Let us take the design R R G R G (meaning red, red, green, red, green). Let the stickers be named Red 1, Red 2, Red 3, Green 1, Green 2. Then each of the following permutations corresponds to the design R R G R G.

Red 1	Red 2	Green 1	Red 3	Green 2
Red 1	Red 3	Green 1	Red 2	Green 2
Red 2	Red 3	Green 1	Red 1	Green 2
Red 2	Red 1	Green 1	Red 3	Green 2
Red 3	Red 1	Green 1	Red 2	Green 2
Red 3	Red 2	Green 1	Red 1	Green 2
Red 1	Red 2	Green 2	Red 3	Green 1
Red 1	Red 3	Green 2	Red 2	Green 1
Red 2	Red 3	Green 2	Red 1	Green 1
Red 2	Red 1	Green 2	Red 3	Green 1
Red 3	Red 1	Green 2	Red 2	Green 1
Red 3	Red 2	Green 2	Red 1	Green 1

All these twelve permutations correspond to the design

Red Red Green Red Green

Solution

There are $5! = 120$ permutations of the five stickers. To each design, there are 12 permutations. Therefore there are $\frac{120}{12} = 10$ designs.

These two examples illustrate the following:

Theorem 12.2

If there are n objects, of which m objects are of one kind, and the remaining $n-m$ objects are of another kind, then the total number of (mutually distinguishable) permutations that can be formed from these is

$$\frac{n!}{m!(n-m)!}$$

Explanation: In Example 12.15, there are 3 toys, so $n = 3$. Of these two are of the same kind. Therefore $m = 2$, our answer was 3. Also, $\frac{3!}{2!(3-2)!} = \frac{6}{2 \times 1} = 3$.

In example 12.16,there are 5 stickers. So $n = 5$. Of these three are red. Therefore, $m = 3$. Our answer was 10. Also,

$$\frac{5!}{3!(5-3)!} = \frac{120}{6 \times 2} = 10.$$

Here the denominator $3!\ (5\text{-}3)! = 12$ is the number of permutations corresponding to one design.

Proof of Theorem 12.2: We know that there are $n!$ permutations, when all the n objects are treated distinct. But now m objects are treated to be of the same kind, and all the remaining to be of another kind.

Fix one particular permutation. If we re-arrange the m objects of the first kind among themselves, and the remaining $n - m$ objects among themselves, the resulting permutation is to be treated as same as the one already fixed. There are $m!(n - m)!$ such.

Thus to each permutation in the collection of $n!$ permutations, there are $m!(n-m)!$ treated as same as that. Therefore there are

$$\frac{n!}{m!(n-m)!} \text{ permutations, mutually distinct.}$$

We state without proof, a more general result.

Theorem 12.3

Let $p_1 + p_2 + \ldots + p_r = n$. Let p_1 objects be of the first kind, p_2 objects be of the second kind, ... and p_r objects be of the rth kind. Then the number of permutations of these n objects is $\frac{n!}{p_1!p_2!\ldots p_r!}$.

The next theorem is merely a reformulation of the above, but considered in a different type of situation.

Theorem 12.4
Suppose there are r objects to be arranged, allowing repetitions. Let further $p_1, p_2, \ldots, p_r$ be the integers such that the first object occurs exactly p_1 times, the second occurs exactly p_2 times, etc. Then the total number of permutations of these r objects subject to the above condition is

$$\frac{(p_1 + p_2 + \ldots + p_r)!}{p_1!p_2!\ldots p_r!}$$

We omit the proof of this also.

Theorem 12.5

The number of permutations of n different objects, taken r at a time, when repetitions are allowed, is n^r.

Proof: The first place can filled up by any one of the n objects. There are n ways to do so.

The second place can be filled up by any one of the n objects because repetition is allowed. There are n ways to do this.

Thus there are $n \times n = n^2$ ways to fill the first two positions.

Similarly, one proves that there are n^r ways to fill up the first r positions.

Circular Permutations

Hitherto, we were considering the arrangements of objects in a line. Suppose we consider arrangements of objects in the form of a circle, instead of a line. Then we speak of circular permutations. (The usual permutations are sometimes called linear permutations.)

Suppose four numbers 1, 2, 3, 4 are to be arranged in the form of a circle.

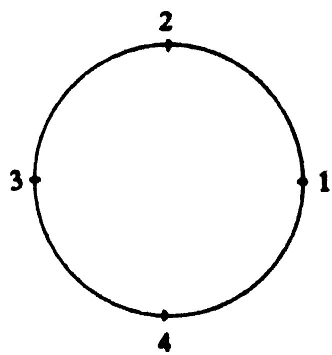


Fig 12.3

One such arrangement is shown in the figure. This arrangement is read in the anticlockwise direction, starting from any point. Thus it may be read as 1234 or 2341 or 3412 or 4123. Thus these four usual permutations correspond to one circular permutation.

It is natural to expect that the number of circular permutations is less than the number of usual (linear) permutations in all nontrivial cases.

Theorem 12.6

The number of circular permutations of n different objects is $(n-1)!$.

Proof: Each circular permutation corresponds to n linear permutations depending on where (out of the n positions) we start. Since there are exactly $n!$ linear permutations, there are exactly $\frac{n!}{n}$ circular permutations. This number is the same as $(n-1)!$.

Example 12.17

In how many ways can 8 students be seated in (i) a circle, (ii) a line?

Solution

The eight students can be seated in a circle in $(8-1)! = 7! = 5040$ ways. They can be arranged in a line in $8! = 40320$ ways.

Example 12.18

How may permutations of the letters of the word APPLE are there?

Solution

Here there are 5 letters, two of which are of the same kind. The others are each of its own kind. Therefore the required number of permutations is

$$\frac{5!}{2!1!1!1!} = \frac{120}{2} = 60$$

Example 12.19

How many words can be formed using the letter A thrice, the letter B twice and the letter C once?

Solution

The six letters given are A, A A, B, B, C. The number of their permutations is

$$\frac{6!}{3!2!1!} = 60$$

Example 12.20

In how many ways can 5 children be arranged in a line such that

(i) two of them, Ram and Shyam, are always together ?

(ii) two of them, Ram and Shyam, are never together ?

Solution

(i) Temporarily, forget Shyam. Consider the other four children. They can be arranged in $4! = 24$ ways. For each such arrangement, Shyam can be placed adjacent to Ram, in two ways, left or right. Therefore, there are totally $24 \times 2 = 48$ ways of arranging as required.

(ii) Among the $5! = 120$ permutations of 5 children, there are 48 in which Ram and Shyam are together. In the remaining $120-48 = 72$ permutations, Ram and Shyam are never together.

Example 12.21

If all permutations of the letters of the word AGAIN are arranged as in a dictionary, what is the fiftieth word?

Explanation: The first word is AAGIN

The second word is AAGNI

We need not write down the full list to find out the fiftieth word.

Solution

Starting with the letter A, and arranging the other four letters, there are $4! = 24$ words. These are the first 24 words. Then starting with G, and arranging A, A, I and N in different ways, there are $\frac{4!}{2!1!1!} = \frac{24}{2} = 12$ words. Next, the 37th word starts with I. There are 12 words starting with I. This accounts up to the 48th word. The 49th word is NAAGI. The 50th word is NAAIG.

EXERCISE 12.5

1. There are three blue balls, four red balls and five green balls. In how many ways can they be arranged in a row?
2. In how many ways can the letters of the word PENCIL be arranged so that N is always next to E?
3. Three boys and three girls are to be seated around a table in a circle. Among them, the boy X does not want any girl neighbour and the girl Y does not want any boy neighbour. How many such arrangements are possible?
4. The principal wants to arrange 5 students on the platform such that the boy SALIM occupies the second position and such that the girl SITA is always adjacent to the girl RITA. How many such arrangements are possible?
5. When a group photograph is taken, all the seven teachers should be in the first row and all the twenty students should be in the second row. If the two corners of the

second row are reserved for the two tallest students, interchangeable only between them, and if the middle seat of the front row is reserved for the principal, how many arrangements are possible?

6. How many even numbers are there with three digits such that if 5 is one of the digits, then 7 is the next digit?
7. A codeword is to consist of two distinct English alphabets followed by two distinct numbers from 1 to 9. For example, $C\ A\ 2\ 3$ is a codeword. How many such codewords are there? How many of them end with an even integer?
8. If there are six periods in each working day of a school, in how many ways can one arrange 5 subjects such that each subject is allowed at least one period?
9. Find the number of permutations of n different things taken r at a time such that two specified things occur together?
10. If the different permutations of the word EXAMINATION are listed as in a dictionary, how many items are there in this list before the first word starting with E?

12.7 Combinations

On many occasions we are not interested in arranging, but only in selecting r objects from n objects. In other words, we do not want to specify the ordering of these selected objects. For example, a student may want to choose three books from his library at a time; a firm may want to recruit 5 of the 10 applicants for a position; an agency may want to select 10 out of 40 students, for awarding scholarships; and so on. In this section we study the methods of counting the number of ways of such selections.

For instance, consider the question: In how many ways can two persons be selected out of four persons? Let A, B, C, D be the four persons. We want to choose two of them. We may choose either A, B, or A, C or A, D or B, C or B, D or C, D. Note that we do not list B, A separately here because it is the same as the choice A,B. Thus there are six ways of selecting 2 persons out of 4 persons.

Notation: We denote the above fact by writing $C(4,2) = 6$. This means there are exactly 6 ways of selecting 2 objects from 4 objects. More generally, $C(n,r)$ denotes the number of ways of selecting r objects from n objects. This makes sense only when $r \leq n$.

It is also customary to denote $C(n,r)$ alternatively by $\binom{n}{r}$ or by ${}^nC_r$.

When we select r objects from n objects, each such selection is called a combination.

Difference Between a Permutation and a Combination

In a combination only selection is made; in a permutation, not only a selection is made, but also an arrangement is there in a definite order.

In a combination, the ordering of the selected objects is immaterial. In a permutation, this ordering is essential. For example A,B and B,A are same as combinations, but different as permutations.

Usually (that is, except in trivial cases) the number of permutations exceeds the number of combinations. For instance, $C(4,2)=6$ and $P(4,2)=12$. The trivial case is when $r=0$ or 1. We have $C(n,0)=1=P(n,0)$ and $C(n,1)=n=P(n,1)$.

In higher mathematics, you will see the combinations $C(n,r)$ occur more often than the permutations.

Each combination corresponds to many permutations. For example, the six permutations 1 2 3, 1 3 2, 2 3 1, 2 1 3, 3 1 2 and 3 2 1 correspond to the same combination 1 2 3

Formula for $C(n,r)$: We have already obtained in section 12.4 a formula for $P(n,r)$ namely,

$$P(n,r)=n(n-1)(n-2)\dots(n-r+1)=\frac{n!}{(n-r)!}$$

Now we obtain a similar formula for $C(n,r)$.

Theorem 12.7
Let n and r be non-negative integers such that $r \leq n$. Then

$$C(n,r)=\frac{n!}{r!(n-r)!}.$$

Proof: For each selection of r objects from n objects, the selected r objects can be arranged in $P(r,r)$ ways. We know $P(r,r)=r!$.Thus for each combination counted in $C(n,r)$, there are $r!$ permutations counted in $P(n,r)$. Therefore

$$C(n,r)=\frac{n!}{r!(n-r)!}$$

Explantion: Sometimes, it is easier to apply it in the form

$$C(n,r)=\frac{n(n-1)\dots(n-r+1)}{1\cdot2\cdot3\dots\cdot r}$$

This is obtained from the above formula by cancelling out $(n-r)!$ in the denominator with the factors $1,2,\dots,n-r$ in the numerator. In order to remember this formula, it is good to note that the numerator is the product of r factors, starting from n and

decreasing one by one, whereas the denominator has the same number of factors starting from 1 and increasing one by one.

Next, the formula $\frac{n!}{r!(n-r)!}$ requires an explanation when $r = n$. Then $(n-r)!$ becomes 0! As already mentioned in section 12.2, the value of 0! is 1.

Thus, $C(n,n) = \frac{n!}{n!0!} = 1$. It is also seen that there is exactly one way of selecting n objects. This gives one more instance to show the consistency of taking 0! as 1.

The formula makes sense even when $r = 0$; we get $C(n,0) = 1$.

Theorem 12.8

Let $0 \leq r \leq n$. Then $C(n,r) = C(n,n-r)$.

Proof :
$$\begin{aligned} C(n,n-r) &= \frac{n!}{(n-r)!(n-(n-r))!} \\ &= \frac{n!}{(n-r)!r!} \\ &= C(n,r) \end{aligned}$$

Explanation: This theorem simplifies the calculation of $C(n,r)$ when r is large. For instance, if we want to calculate $C(10,9)$, this theorem says that

$C(10,9) = C(10,10-9) = C(10,1)$. It is easier to compute $C(10,1)$ as 10.

The Theorem can be restated as follows:

If p and q are non-negative integers such that $p+q=n$, then $C(n,p) = C(n,q)$.

Theorem 12.9

Let n and r be non-negative integers such that $r \leq n$. Then $C(n,r) + C(n,r-1) = C(n+1,r)$.

Proof :
$$\begin{aligned} C(n,r) + C(n,r-1) &= \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)!} \\ &= \frac{n!}{r!(n-r+1)!}[(n-r+1)+r] \\ &= \frac{n!(n+1)}{r!(n-r+1)!} = \frac{(n+1)!}{r!(n+1-r)!} \\ &= C(n+1,r). \end{aligned}$$

Remark

Such identities are called combinatorial identities. They can be alternatively proved by combinatorial arguments also. These proofs will be indicated in section 12.9.

Example 12.22

If $C(n,7) = C(n,5)$, find n.

Solution

We know that if p and q are non-negative integers such that $p+q=n$, then $C(n,p) = C(n,q)$. Here taking $p=7$ and $q=5$, it follows that

$$C(12,7) = C(12,5)$$

Therefore, $n = 12$ is our answer.

But this method does not answer the question. Are there other values of n satisfying $C(n,7) = C(n,5)$? In fact, there is no other. This can be argued as follows:
The given equation $C(n,7) = C(n,5)$ means $\frac{n!}{7!(n-7)!} = \frac{n!}{5!(n-5)!}$. This gives $7!(n-7)! = 5!(n-5)!$
After cancelling the factors common to both sides, this gives $6 \times 7 = (n-6)(n-5)$. This yields the quadratic equation $n^2 - 11n - 12 = 0$. Its roots are $n = 12$ and $n = -1$. In this problem n should be non-negative. Therefore, $n = 12$ is the only solution.

Example 12.23

Prove that $nC(n-1, r-1) = (n-r+1)C(n, r-1)$ for all $1 \leq r \leq n$.

Solution

$$\begin{aligned}
\text{Left hand side} &= nC(n-1,r-1) \\
&= \frac{n(n-1)!}{(r-1)!\,((n-1)-(r-1))!} \\
&= \frac{n!}{(r-1)!(n-r)!} \\
\text{Right hand side} &= (n-r+1)C(n,r-1) \\
&= \frac{(n-r+1)n!}{(r-1)!(n-r+1)!} \\
&= \frac{n!}{(r-1)!(n-r)!}
\end{aligned}$$

Thus both sides are found to be equal to the same quantity.

EXERCISE 12.6

1. Calculate $C(10,8)$.
2. Verify the equality $2C(7,4) = C(8,4)$.
3. Similar to the above problem, do we have $2C(8,4) = C(9,4)$?
4. If $C(n,8) = C(n,6)$, find $C(n,2)$.

5. If the ratio $C(2n,3) : C(n,3)$ is equal to 11:1, find n.
6. Prove that $C(2,1) + C(3,1) + C(4,1) = C(3,2) + C(4,2)$.
7. Prove that $\sum\limits_{r=1} C(5,r) = 31$.
8. Prove that $1 + C(3,1) + C(4,2) = C(5,3)$.

12.8 Practical Problems on Combinations

In this section, we solve some problems in actual life where the formula for $C(n,r)$ can be applied.

Example 12.24
In how many ways can 5 sportsmen be selected from a group of 10?

Solution
The required number is $C(10,5) = \frac{10!}{5!5!} = 252$.

Example 12.25
If there are 12 persons in a party, and if each two of them shake hands with each other, how many handshakes happen in the party? (Explanation: When two persons shake hands, it is counted as one handshake, not two. Therefore this is a problem on combinations, not permutations.)

Solution
The total number of handshakes is the same as the number of ways of selecting 2 persons from among 12 persons. This is $C(12,2) =$

Example 12.26
A student has to answer 10 questions, choosing at least 4 from each of Part A and Part B. If there are 6 questions in part A and 7 in Part B, in how many ways can the student choose 10 questions?
Explanation: In this problem, after choosing 10 questions, the student may answer them in any order. We do not want to count the number of these arrangements of the chosen 10 questions. We want to count only the number of ways of choosing them.

Solution
The possibilities are:

4 from Part A and 6 from Part B
or 5 from Part A and 5 from Part B
or 6 from Part A and 4 from Part B

Therefore the required number of ways is

$$C(6,4)C(7,6) + C(6,5)C(7,5) + C(6,6)C(7,4)$$

$$= \frac{6 \times 5}{1 \times 2} \times \frac{7}{1} + \frac{6}{1} \times \frac{7 \times 6}{1 \times 2} + \frac{7 \times 6 \times 5}{1 \times 2 \times 3}$$

$$= 105 + 126 + 35 = 266$$

Explanation: Here we have used $C(7,6) = C(7,1)$ etc.

EXERCISE 12.7

1. From a class of 32 students, 4 are to be chosen for a competition. In how many ways can this be done?
2. A question paper has two parts, Part A and Part B, each containing 10 questions. If the student has to choose 8 from Part A and 5 from Part B, in how many ways can he choose the questions?
3. A boy has 3 library tickets and 8 books of his interest in the library. Of these 8, he does not want to borrow Chemistry Part II, unless Chemistry Part I is also borrowed. In how many ways can he choose the three books to be borrowed?
4. A sports team of 11 students is to be constituted, choosing at least 5 from class XI and at least 5 from class XII. If there are 20 students in each of these classes, in how many ways can the teams be constituted?
5. From a class of 25 students, 10 are to be chosen for an excursion party. There are 3 students who decide that either all of them will join or none of them will join. In how many ways can they be chosen?
6. From a class of 12 boys and 10 girls, 10 students are to be chosen for a competition, at least including 4 boys and 4 girls. The 2 girls who won the prizes last year should be included. In how many ways can the selection be made?
7. In a village, there are 87 families, of which 52 families have atmost 2 children. In a rural development programme, 20 families are to be helped chosen for assistance, of which at least 18 families must have atmost 2 children. In how many ways can the choice be made?
8. If 20 lines are drawn in a plane such that no two of them are parallel and no three are concurrent, in how many points will they intersect each other?
9. How many different products can be obtained by multiplying two or more of the numbers 3,5,7,11 (without repetition)?
10. In a certain city, all telephone numbers have six digits, the first two digits always being 41 or 42 or 46 or 62 or 64. How many telephone numbers have all six digits distinct?

12.9 Combinatorial Identities and Arguments

There are certain identities or equations involving the symbols $P(n,r)$ and $C(n,r)$. Here we list some of them, including the ones that we have already seen:

1. $P(n,r) = r!C(n,r)$
2. $P(n,n) = P(n,n-1)$
3. $P(n,n) = nP(n-1,n-1)$
4. $(n+1)P(n,r) = (n-r+1)P(n+1,r)$
5. $P(n,n) = r!P(n,n-r)$
6. $P(n,r) = P(n-1,r) + rP(n-1,r-1)$
7. $C(n,r) + C(n,r-1) = C(n+1,r)$
8. $rC(n,r) = nC(n-1,r-1)$
9. $C(n,r) = C(n,n-r)$
10. $C(n,0) + C(n,1) + C(n,2) + \ldots + C(n,n) = 2^n$

All these can be proved, using the formulae for $C(n,r)$ and $P(n,r)$.

But many of them can be proved by other arguments as well. Let us see some examples.

Example 12.27

Prove that $P(n,r) = P(n-1,r) + rP(n-1,r-1)$.

Solution

Consider n objects, out of which r are to be taken and arranged. Among these n objects, fix one of them for the present, and consider the two cases: those permutations in which this fixed object is not taken, and those in which it is taken.

(There are exactly $P(n-1,r)$ permutations of the first type. We shall prove that there are $rP(n-1,r-1)$ permutations of the second type. Now every permutation of the second type contains the fixed object at some position. How many of them have this fixed object as the first position? There are exactly $P(n-1,r-1)$ of them. Similarly, there are exactly $P(n-1,r-1)$ permutations in which this fixed object is in the second position.

Thus proceeding, since there are exactly r positions, we obtain that there are exactly $rP(n-1,r-1)$ permutations of the second type. Thus among the $P(n,r)$ permutations of n objects, taken r at a time, there are $P(n-1,r)$ that are of the first type, and $rP(n-1,r-1)$ that are of the second type. Therefore
$P(n,r) = P(n-1,r) + rP(n-1,r-1)$.

Example 12.28
Prove that $C(n,r) = C(n,n-r)$.

Solution
For each way of our choosing r objects out of the given n objects, imagine that there is an opponent who chooses the remaining $n-r$ objects out of n objects. We notice that each choice of $n-r$ objects out of n objects, can arise in this way. Therefore there are as many ways of (our) choosing r objects out of n objects, as there are ways of (opponent's) choosing $(n-r)$ objects out of n objects. Therefore $C(n,r) = C(n,n-r)$.

Example 12.29
Give a set-theoretic argument to prove $C(n,r) = C(n,n-r)$.

Solution
Let A be the set $\{1,2,3,\ldots,n\}$. How many subsets of A are there, containing exactly r elements? This number is precisely $C(n,r)$.

Whenever B is a subset of A having r elements, it is true that the complement $A-B$ is a subset of A having $n-r$ elements where $A-B$ means the set of those elements of A which are not in B.

Also, every subset of A having $n-r$ elements is from $A-B$ where B is a subset of A having r elements.

Thus the number of subsets of A having r elements is the same as the number of subsets of A having $n-r$ elements.

In other words, $C(n,r) = C(n,n-r)$.

Example 12.30
Prove by combinatorial argument that

$$C(n+1,r) = C(n,r) + C(n,r-1)$$

Solution
Consider all possible choices of r objects from a given collection of $n+1$ objects. There are $C(n+1,r)$ such choices.

Now fix one object, say A and use it to divide these choices into two types: Those that include A are of the first type, those that do not include A are of the second type.

Every choice of first type, includes A and $r-1$ other elements chosen from the n objects that are other than A. Thus there are $C(n,r-1)$ choices of first type.

Every choice of second type, has r elements chosen from the n objects that are other than A. Thus there are $C(n,r)$ choices of the second type.

Summing the two, there are $C(n,r)+C(n,r-1)$ choices of both the types. This proves that $C(n+1,r)$ must be equal to this number.

EXERCISE 12.8

Give combinatorial arguments for the following identities:

1. $P(n,n)=P(n,n-1)$.

   (*Hint:* For each permutation of n objects taken all at a time, a permutation of n objects, taking $n-1$ at a time, can be associated by omitting the last object. Conversely, each permutation of n objects, taken $n-1$ at a time, determines a unique permutation of n objects, taken all at a time, obtained by adding the omitted object at the end.)

2. $P(n,n)=nP(n-1,n-1)$.

   (*Hint:* Fix an object A from among the given n objects. Every permutation on these n objects is uniquely determined by a permutation on the $n-1$ objects that are other than A and by the position of A therein.)

3. $C(n,r)P(r,r)=P(n,r)$.

   (*Hint:* Every permutation of n objects, taken r at a time is uniquely determined by first choosing r objects out of these n objects and then by arranging these r objects.)

4. $P(n,n)=P(r,r)P(n,n-r)$.

   (*Hint:* Fix r objects among the given n objects. Every permutation of n objects is determined by first choosing the r blank spaces from among n of the blank spaces, by next arranging the fixed r objects in these r blank spaces, and then by arranging the remaining $n-r$ objects in the remaining $n-r$ blank spaces. Also, use the previous identity.)

5. $rC(n,r)=nC(n-1,r-1)$.

   (*Hint:* Consider the number of ways of choosing r objects from n objects, and then choosing one of the already chosen r objects.)

6. $C(n,0)+C(n,1)+C(n,2)+\ldots+C(n,n)=2^n$.

   (*Hint:* Let A be a set having n elements. Then $C(n,r)$ is the number of subsets of A having exactly r elements. Every subset is uniquely determined by answering the question for each of the n elements, whether that element is or is not in that subset.)

7. $C(n,r)C(r,s) = C(n,s)C(n-s, r-s)$.

   (*Hint:* First choose r objects from n objects, and then choose s objects from the already chosen r objects.)

8. $(n-r)C(n,r) = (r+1)C(n,r+1)$.

   (*Hint:* Each choice of r objects from n objects is obtained in $n-r$ ways, by choosing $r+1$ objects from n objects and then omitting one of them.)

9. $C(n,n) = 1$.

10. $C(n,2) = \frac{1}{2}n(n-1)$.

11. Prove all the above identities by using the formulas for $C(n,r)$ and $P(n,r)$.

MISCELLANEOUS EXERCISE ON CHAPTER 12

1. In how many ways can a football team of 11 players be selected from 16 players? How many of these will

   (i) include 2 particular players?

   (ii) exclude 2 particular players?

2. A committee of 5 is to be selected from amongst 6 boys and 5 girls. Determine the number of ways of selecting the committee if it is to consist of atleast one boy and one girl.

3. In an examination, a student has to answer 4 questions out of 5 questions; questions 1 and 2 are however compulsory. Determine the number of ways in which the student can make the choice.

4. Find the number of diagonals of a hexagon.

5. A polygon has 44 diagonals. Find the number of its sides.

ANSWERS

EXERCISE 7.1

1. (i) $(0,1)$, $\sqrt{2}$ (ii) $(-5,3)$, $\sqrt{30}$
   (iii) $(\frac{1}{2}, -\frac{1}{3})$, $\frac{1}{2}$ (iv) $(2,-3)$, $3\sqrt{2}$ (v) $(\frac{1}{2}, -1)$, $\frac{\sqrt{11}}{2}$

2. (i) $36x^2 + 36y^2 - 36x - 18y + 11 = 0$
   (ii) $x^2 + y^2 + 6x + 4y - 36 = 0$
   (iii) $x^2 + y^2 + 2y = 0$
   (iv) $x^2 + y^2 - x - y = 0$
   (v) $x^2 + y^2 - 2ax - 2ay = 0$
   (vi) $x^2 + y^2 - 2ax\cos\alpha - 2ay\sin\alpha = 0$

3. $x^2 + y^2 + 4x - 21 = 0$ and $x^2 + y^2 - 12x + 11 = 0$

EXERCISE 7.2

1. (i) $x^2 + y^2 + 6x + 2y - 90 = 0$
   (ii) $x^2 + y^2 - 2x + 6y - 40 = 0$
   (iii) $13x^2 + 13y^2 - 64x + 10y - 332 = 0$
   (iv) $x^2 + y^2 + 4x + 6y - 12 = 0$

2. (a) $x^2 + y^2 - ax - by = 0$
   (b) $x^2 + y^2 - 2hx - 2ky = p^2 + q^2 - 2hp - 2kq$

3. $x^2 + y^2 - 4x - 2y - 20 = 0$; $(2,1)$, 5.

4. $x^2 + y^2 - 4x - 6y - 87 = 0$

EXERCISE 7.3

1. (i) $x = \frac{2}{\sqrt{3}}\cos\alpha,\ y = \frac{2}{\sqrt{3}}\sin\alpha$
   (ii) $x = -1 + 3\cos\alpha,\ y = 2 + 3\sin\alpha$
   (iii) $x = -\frac{p}{2} + \frac{p}{\sqrt{2}}\cos\alpha,\ y = -\frac{p}{2} + \frac{p}{\sqrt{2}}\sin\alpha$

2. (i) $x^2+y^2=9; (0,0), 3$
   (ii) $(x-a)^2+(y-b)^2=c^2; (a,b), c$
   (iii) $(x-7)^2+(y+3)^2=16; (7,-3), 4$
   (iv) $4x-y-5=0$

EXERCISE 7.4

1. $x^2+y^2-(p+r)x-(q+s)y+pr+qs=0$

2. $x^2+y^2-3x-2y-21=0$

EXERCISE 7.5

1. (3, – 4) and (4, 3)

2. $\left[\dfrac{-mc\pm\sqrt{a^2(1+m^2)-c^2}}{1+m^2}, \dfrac{c\pm m\sqrt{a^2(1+m^2)-c^2}}{1+m^2}\right]; a^2(1+m^2)=c^2$

3. (0,2) and (2,0)

4. ± 5

EXERCISE 7.6

1. (i) $x+y=2$
   (ii) $x-3y=10$
   (iii) $3x+4y=-25$
   (iv) $11x-2y=46$
   (v) $11x-2y=16$
   (vi) $4x+3y+6=0$
   (vii) $x\cos\alpha+y\sin\alpha=a(1+\cos\alpha)$

3. $a^2(l^2+m^2)=n^2$

4. $2x+y\pm 3\sqrt{5}=0$

5. $\frac{1}{2}\sqrt{46}$

6. $4\sqrt{3}$

8. $y=x\sqrt{3}\pm 2\sqrt{3}$

EXERCISE 7.7

1. $x^2+y^2-4x+2y+(5-r^2)=0,\qquad x^2+y^2-4x+2y-4=0$

2. $x^2+y^2+8x-4y+(20-r^2)=0,\qquad 2x^2+2y^2+16x-8y-41=0$

3. $x^2+y^2-6x+10y+(34-r^2)=0,\qquad x^2+y^2-6x+10y+9=0$

4. (i) $6x^2+6y^2-44x+43=0$
   (ii) $23x^2+23y^2-156x+38y+168=0$

5. $x^2+y^2-(3k-4)\,x-2ky+\dfrac{(3k-4)^2}{4}+k^2-r^2=0$

EXERCISE 7.8

1. $3x^2+3y^2-14x+23y-15=0$

EXERCISE 8.1

1. (a) $(2,0), x=-2$ (b) $(0,\frac{3}{2}), y=-\frac{3}{2}$
   (c) $(-3,0), x=3$ (d) $(0,-4,), y=4$

2. (a) $y^2=-16x$ (b) $x^2=-8y$ (c) $2y^2=9x$

3. (a) $(1,\frac{9}{4}),(1,2), y=\frac{7}{4}, x=1$
   (b) $(\frac{3}{8},\frac{9}{16}),\ (\frac{3}{8},\frac{1}{2}),\ y=\frac{5}{8}, x=\frac{3}{8}$
   (c) $(\frac{3}{2},-\frac{15}{8}),(\frac{3}{2},-\frac{11}{8}),\ y=-\frac{7}{8}, x=\frac{3}{2}$

4. (a) $y^2=12x-36$ (b) $x^2=32-8y$
   (c) $4x^2+4xy+y^2+4x+32y+16=0$

EXERCISE 8.2

1. (a) $10,\ 8,\ (\pm3,0),(\pm5,0),\frac{3}{5}$ (b) $8,6,(\pm\sqrt{7},0),(\pm\ 4,\ 0),\ \frac{\sqrt{7}}{4}$
   (c) $2\sqrt{3},2\sqrt{2},(0,\pm1),(0,\pm\sqrt{3}),\frac{1}{\sqrt{3}}$ (d) $2,1,(1\pm\frac{\sqrt{3}}{2},0),(2,0),(0,0),\frac{\sqrt{3}}{2}$

2. (a) $9x^2+25y^2=225$ (b) $100x^2+36y^2=3600$
   (c) $25x^2+9y^2=225$ (d) $7x^2+15y^2=247$
   (e) $x^2+2y^2=18$ (f) $16x^2+7y^2=688$

3. $3x^2+4y^2-36x=0$

4. $9x^2+5y^2=180$

EXERCISE 8.3

1. (a) $6, 8, \frac{5}{3}, (\pm 5, 0), (\pm 3, 0)$

   (b) $2\sqrt{3}, 2\sqrt{2}, \sqrt{\frac{5}{3}}, (\pm\sqrt{5}, 0), (\pm\sqrt{3}, 0)$

   (c) $\frac{2}{\sqrt{3}}, \sqrt{2}, \sqrt{\frac{5}{2}}, (\pm\sqrt{\frac{5}{6}}, 0), (\pm\frac{1}{\sqrt{3}}, 0)$

2. (a) $24x^2 - 25y^2 = 600$ (b) $9x^2 - 7y^2 - 343 = 0$ (c) $y^2 - x^2 = 5$

3. $15x^2 - y^2 = 15$

4. $2\sqrt{3}, 8, \sqrt{\frac{19}{3}}$

EXERCISE 8.4

3. $(2,4)$

5. $yt = x + at^2$

6. $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$

7. $y = 3x \pm \frac{31\sqrt{20}}{4\sqrt{93}}, (\mp\frac{9}{4}\cdot\frac{\sqrt{20}}{\sqrt{93}}, \pm\frac{\sqrt{20}}{\sqrt{93}})$

8. $24y = 30x \pm \sqrt{161}$

Miscellaneous Exercise on Chapters 4, 5, 6, 7 and 8

2. $(7, 5), (-1, -1)$

3. $p^2(x^2 + y^2) = 4x^2y^2$

4. $3x + y + 7 = 0$ or $x - 3y - 31 = 0$

7. $x^2 + y^2 \pm 10x - 8y + 16 = 0$

8. $4x + 3y + 19 = 0$ and $4x + 3y - 31 = 0$

9. $x^2 + y^2 - 16x - 18y - 4 = 0$

10. $4x^2 + y^2 - 4xy - 72x - 64y + 24 = 0$

11. $(1, \frac{3}{2})$

13. $7x^2 - 2y^2 + 12xy - 2x + 14y - 22 = 0$

15. $(1,1)$

EXERCISE 9.1

3.

EXERCISE 9.2

1. No, For example $|2+3i| = |3+2i|$ but $2+3i \neq 3+2i$

Miscellaneous Exercise on Chapter 9

5. (i) $8+i$ (ii) $\frac{11}{17} - \frac{10i}{17}$

6. 2

7. $1-4i,\ -1+4i$

9. The relation $\sqrt{ab} = \sqrt{a}\sqrt{b}$, assumed here, is valid only when a,b are nonnegative real numbers.

13. (a) 1 (b) w^2 (c) 1 (d) 1

15. (i) $\frac{22}{17} + \frac{3i}{17}$ (ii) $\frac{3}{2} + \frac{1}{2}i$

16. (i) $\frac{3}{13} - \frac{2i}{13}$ (ii) $\frac{2}{7} - \frac{\sqrt{3}}{7}i$

17. (a) $\pm(4-3i)$
(b) $\pm(3-2i)$
(c) $\pm(1+\sqrt{3}i)$

EXERCISE 10.1

1. $x^2 - 5x + 6 = 0$

2. (i) $-\frac{1}{3}$ (ii) $\frac{-32}{9}$ (iii) $3\frac{19}{27}$

3. 4

4. 1,5

5. $a = 4$

6. $10m$

7. Rs 180

8. $2,\ 3,\ \frac{5 \pm \sqrt{3}\,i}{2}$

9. $\pm a$

10. $\frac{3}{5}, \frac{3}{5}$

11. $10(\sqrt{2}-1)\%$

12. 5

13. 3 and 15 or -15 and -3

14. -16

15. $\frac{9}{2}$

16. $-1,\ 2$

17. $\frac{9}{13}, \frac{4}{13}$

18. 0, 3

EXERCISE 11.1

1. $a_1 = 33, d = -4,$ 3. 19 4. $\dfrac{cr - dr + dp - cq}{p - q}$ 5. 21, 23, 25

7. $\dfrac{4}{5}$ 9. n 10. 1150 11. 4,8

EXERCISE 11.2

2. 6, 9, 12, 15, 18, 21 3. -10

EXERCISE 11.3

1. $3, -6, 12, -24$ 2. ± 3 3. $a = \dfrac{-4}{3}$, $r = 2$ or $a = 4, r = -2$ 4. 8, 12, 18

8. $26, 5, -16$ or $2, 5, 8$

EXERCISE 11.4

1. $\dfrac{100}{19}$ 2. $\dfrac{2}{3}$

EXERCISE 11.5

1. $4\dfrac{8}{9}$ 2. $\dfrac{1}{4}$ or $\dfrac{17}{11}$ 3. 2 4. $\left(\dfrac{3}{2}\right)^2 \left(1 - \dfrac{1}{3^n}\right) - \dfrac{3n}{2} \cdot \dfrac{}{3}$

EXERCISE 11.6

1. $\dfrac{2n(n+1)(2n+1)}{3}$ 2. $n^2(2n^2 - 1)$

MISCELLANEOUS EXERCISE ON CHAPTER 11

2. Converse is not true

3. 6. **4.** $-4, -1, 2$ **5.** 852 **6.** 9 hours **7.** Never

8. ± 3 **9.** 8, 16, 32 **10.** $2^{64} - 1$ **15.** 6 **17.** 14 **18.** $\frac{31}{48}$

20. 12 **21.** 2, 4, 6, 8 **22.** $\frac{7}{81}(4490 + 10^{-49})$

23. 4 **24.** (i) $\frac{50}{81}(10^n - 1) - \frac{5n}{9}$ n, (ii) $\frac{2}{3}n - \frac{2}{27}(1 - 10^{-n})$

25. 2187 or $\frac{2187}{5}$ **26.** 4, 64 **27.** $\frac{2}{9}$

28. (i) $\frac{1}{4}n(n+1)(n+2)(n+3)$ (ii) $\frac{1}{12}n(n+1)(n+2)(3n+5)$

EXERCISE 12.1

1. 120 **2.** 1944 **3.** 64

4. 648 **5.** 15600 **6.** 24 **7** 6

8. 14 **9.** 31 **10.** 8 **11.** 2560000

EXERCISE 12.2

1. 870 **2.** None of them is true. **3.** $x = 121$

4. LCM is 6! **5.** 3 **6.** 60 and 10

7. 5 **10.** (i) $\frac{11!}{3!}$, (ii) $2^5 \cdot 5!$ **11.** 3 **13.** 31

EXERCISE 12.3

5. 3 **6.** 3

8. 9 **9.** 9

EXERCISE 12.4

1. 24 **2.** 120 **3.** 24 **4.** 720

5. 12 **6.** $8!$ **7.** 4536 **8.** $2(16!)^2$

9. 60 **10.** 1260

EXERCISE 12.5

1. 27720 **2.** 120 **3.** 4 **4.** 8

5. $18!$ (1440) **6.** 365 **7.** 46800; 20800 **8.** 3600

9. $2(r-1)\,P(n-2, r-2)$ **10.** 907200

EXERCISE 12.6

1. 45 **3.** No **4.** 91 **5.** 6

EXERCISE 12.7

1. 35960 **2.** 11340 **3.** 41

4. $2C(20, 5)\,C(20, 6)$

5. 817190

6. 104874

7. $C(52, 18)\,C(35, 2) + C(52, 19)\,C(35, 1) + C(52, 20)$

8. 190 **9.** 11 **10.** 8400

MISCELLAENEOUS EXERCISE ON CHAPTER 12

1. 4368 (i) 2002 (ii) 364 **2.** 455 **3.** 3 **4.** 9 **5.** 11